Preparatory course for M.Sc. Mathematical Physics (Oct. 2020)

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Contents: 7 video lectures

+ 7 question + exercise sessions

1) Topological, metric, and normed spaces
2) Continuity, compact sets, connected sets
3) Differential calculus
4) Ordinary differential equations
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Lect. 1: Topological, metric, and normed spaces

Convergence:

A sequence $(x_n) = (x_1, x_2, x_3, \dots)$ in a set X is a map

$$x : \mathbb{N} \to X, \quad n \mapsto x_n$$

Def.: A sequence (x_n) in $\mathbb{R}$ converges to $x \in \mathbb{R}$, iff

$$\forall \varepsilon > 0 \ \ \exists N_\varepsilon \in \mathbb{N} \ \ \forall n \geq N_\varepsilon : \ |x_n - x| < \varepsilon.$$

(iff for every $\varepsilon > 0$ it holds „eventually" that $|x_n - x| < \varepsilon$)

(x_n) in $\mathbb{R}^n$: — " — : $\|x_n - x\| < \varepsilon$

(x_n) in a metric space: — " — : $d(x_n, x) < \varepsilon$

$:\Leftrightarrow x_n \in B_\varepsilon(x)$

(x_n) in a topol. space: iff for every neighborhood U of x, eventually $x_n \in U$.

Def.: Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$.

A norm $\|\cdot\|$ on V is a map $\|\cdot\| : V \to [0, \infty)$, $x \mapsto \|x\|$

with the properties:

(i) $\|x\| = 0 \iff x = 0$

(ii) $\forall x \in V, \lambda \in \mathbb{K} : \ \|\lambda x\| = |\lambda| \cdot \|x\|$

(iii) $\forall x, y \in V : \ \|x + y\| \leq \|x\| + \|y\|$

The pair $(V, \|\cdot\|)$ is called a normed space.

Examples: a) On $V = \mathbb{R}^n$ or $\mathbb{C}^n$ the foll. maps are norms:

$$\|x\|_2 := \sqrt{|x_1|^2 + |x_2|^2 + \dots + |x_n|^2} \quad \text{(euclidean norm)}$$

$$\|x\|_\infty := \max\{|x_1|, \dots, |x_n|\} \quad \text{(maximum norm)}$$

$$\|x\|_1 := |x_1| + |x_2| + \dots + |x_n| \quad \text{(1-norm)}$$

or, more generally, for $p \in [1, \infty)$

$$\|x\|_p := \left(\sum_{j=1}^n |x_j|^p \right)^{1/p} \quad \text{(p-norm)}$$

b) Let X a set, $(Y, \|\cdot\|_Y)$ a normed space,

and
$$V := \{ f : X \to Y \mid \sup_{x \in X} \|f(x)\|_Y < \infty \}.$$

Then
$$\|f\|_\infty := \sup_{x \in X} \|f(x)\|_Y$$
is a norm on V. (Exercise!: Show this!)

Def.: Let X be a set. A metric d on X is a map
$$d : X \times X \to [0, \infty)$$
with the following properties:

(i) $d(x,y) = 0 \iff x = y$

(ii) $\forall x, y \in X : d(x,y) = d(y,x)$

(iii) $\forall x, y z \in X : d(x,z) \le d(x,y) + d(y,z)$

The pair (X, d) is called a metric space.

Examples: a) Let $(V, \|\cdot\|)$ be a normed space. Then

$d : V \times V \to [0, \infty), \ (x, y) \mapsto d(x,y) := \|x - y\|$

defines a metric on V.

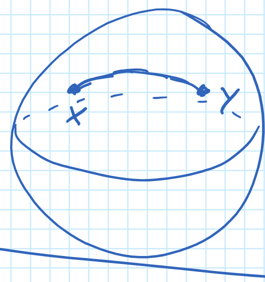
b) Let X be a set. Then the discrete metric on X is

$$d(x,y) := \begin{cases} 0 & \text{if } x = y \\ 1 & \text{otherwise} \end{cases}$$

c) The euclidean unit sphere $S^2 := \{ x \in \mathbb{R}^3 \mid \|x\|_2 = 1 \}$ with the metric

$$d(x,y) := \arccos(\langle x, y \rangle)$$

is a metric space.



Def.: (Open sets in metric spaces)

Let (X, d) be a metric space.

(a) For $x_0 \in X$ and $r > 0$ the set

$$B_r(x_0) := \{ x \in X \mid d(x, x_0) < r \}$$

is called the open ball with radius r and center x_0.

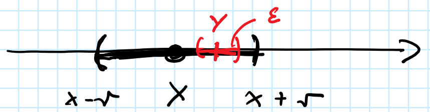
(b) A subset $U \subset X$ is called a neighborhood of $x_0 \in X$, iff U contains an open ball around x_0, i.e. $\exists r > 0 : B_r(x_0) \subset U$.

Then x_0 is called an interior point of U.

(c) A subset $U \subset X$ is called open, iff it contains

(c) A subset $U \subset X$ is called <u>open</u>, iff it contains only interior points, i.e. iff $\forall x \in U \ \exists r > 0 : B_r(x) \subset U$.

<u>Exampls:</u> a) Let (X, d) be a metric space. Then for any $x \in X$ and $r > 0$ the set $B_r(x)$ is open. (Exe. 2: Prove this!)



$\forall y \in B_r(x) \ \exists \varepsilon > 0$

$(y-\varepsilon, y+\varepsilon) \subset (x-r, x+r)$

b) Let X be equipped with the discrete metric.

Then any subset $U \subset X$ is open: $B_{\frac{1}{2}}(x) = \{x\} \ \forall x \in X$.

<u>Prop.:</u> Let (X, d) be a metric space. Then

(i) $\emptyset$ and X are open.

(ii) If $U, V \subset X$ are open, then also $U \cap V$ is open.

(iii) If $U_i \subset X$ is open for all $i \in I$, then also $\bigcup_{i \in I} U_i$ is open. <u>Exe. 3:</u> Prove this!

(ii) implies that intersections of finitely many open sets are open. However, this is not true for infinitely many sets: Let $U_n := (-\frac{1}{n}, \frac{1}{n}) \subset \mathbb{R}$, $n \in \mathbb{N}$.

Then $\bigcap_{n \in \mathbb{N}} U_n = \{0\}$ is not open.

Def.: A subset $A \subset X$ of a metric space is closed

Def.: A subset $A \subset X$ of a metric space is closed, iff its complement $A^c := \{x \in X \mid x \notin A\}$ is open.

Examples: Let $X = \mathbb{R}$ and $a, b \in \mathbb{R}$ with $a < b$.

Then $[a,b]$, $[a,\infty)$ are closed, but $[a,b)$ is neither open nor closed.

For any metric space (X,d) the sets $\emptyset$ and X are open and closed.

Def.: Let X be a set. A topology $\mathcal{T}$ on X is a collection $\mathcal{T} \subset \mathcal{P}(X)$ of subsets of X with the following properties:

(i) $\emptyset, X \in \mathcal{T}$

(ii) $U, V \in \mathcal{T} \Rightarrow U \cap V \in \mathcal{T}$

(iii) $U_i \in \mathcal{T}$ for all $i \in I \Rightarrow \bigcup_{i \in I} U_i \in \mathcal{T}$

The sets $U \in \mathcal{T}$ are called the open sets and $(X, \mathcal{T})$ is called a topological space.

$A \subset X$ is closed, iff $A^c \in \mathcal{T}$.

For $U \subset X$ is called a neighbourhood of $x_0 \in X$ (and x_0 an interior point of U), iff $\exists\, O \in \mathcal{T}$

(and x_0 an interior point of $\mathcal{U}$), iff $\exists\, \mathcal{O} \in \mathcal{T}$ with $x_0 \in \mathcal{O} \subset \mathcal{U}$.

Example: According to the above Prop., the open sets in a metric space form a topology.

Def: Let $(X, \mathcal{T})$ be a top. space and $Y \subset X$.

(a) The set $\mathring{Y} := \bigcup_{\substack{\mathcal{U} \in \mathcal{T} \\ \mathcal{U} \subset Y}} \mathcal{U}$ is called the interior of Y

(b) The set $\overline{Y} := \bigcap_{\substack{\mathcal{U} \in \mathcal{T} \\ \mathcal{U} \subset Y^c}} \mathcal{U}^c$ is called the closure of Y

(c) The set $\partial Y := \overline{Y} \setminus \mathring{Y}$ is called the boundary of Y.

By definition we have

- $\mathring{Y} \subset Y \subset \overline{Y}$
- $\mathring{Y}$ is the largest open set contained in Y.
- Y is open $\Leftrightarrow Y = \mathring{Y}$
- $\overline{Y}$ is the smallest closed set containing Y.
- Y is closed $\Leftrightarrow Y = \overline{Y}$
- $(\mathring{Y})^c = \overline{(Y^c)}$ and $(Y^c)^\circ = (\overline{Y})^c$ (de Morgan)

Prop.: Let $(X, \mathcal{T})$ be a top. space and $Y \subset X$. Then

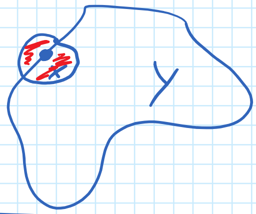
Prop.: Let $(X, \mathcal{T})$ be a top. space and $Y \subset X$. Then

(a) $\mathring{Y}$ is the set of interior points of Y.

(b) $x \in \partial Y \iff$ for any neighborhood U of x

$$U \cap Y \neq \emptyset \text{ and } U \cap Y^c \neq \emptyset.$$

(c) $\mathring{Y} = Y \setminus \partial Y$ and $\overline{Y} = Y \cup \partial Y$



Example: For $Y = [a, b) \subset \mathbb{R}$ we have

$$\mathring{Y} = (a, b), \quad \overline{Y} = [a, b], \quad \partial Y = \{a, b\}.$$

For $\mathbb{Q} \subset \mathbb{R}$ we have $\mathring{\mathbb{Q}} = \emptyset$, $\overline{\mathbb{Q}} = \mathbb{R}$, $\partial \mathbb{Q} = \mathbb{R}$.

Def.: Let X be a top. space. A sequence (x_n) in X converges to $a \in X$ and we write

$$\lim_{n \to \infty} x_n = a,$$

iff for any neighborhood U of a there exist $N \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq N$.

Note that in general convergence points are not unique: on any set X with the indiscrete topology $\mathcal{T} := \{\emptyset, X\}$ any sequence (x_n) in X converges to every point in X!

Def.: A topological space $(X, \mathcal{T})$ is Hausdorff, iff $\forall x, y \in X,\ x \neq y,\ \exists U, V \in \mathcal{T}: x \in U, y \in V, U \cap Y = \emptyset$.

In Hausdorff spaces sequences have at most one limit.

Metric spaces are Hausdorff: for $x, y \in X$, $x \neq y$, we have $d(x,y) =: r > 0$ and $B_{\frac{r}{2}}(x) \cap B_{\frac{r}{2}}(y) = \emptyset$.

Def.: A point $a \in X$ is called a cluster point of a sequence (x_n) if any neighbourhood U of a contains infinitely many elements of (x_n).

Prop.: Let X be a metric space and $A \subset X$. Then

$$a \in \overline{A} \iff \exists (x_n) \text{ in } A : \lim_{n \to \infty} x_n = a$$

Note that $\Leftarrow$ holds also in top. spaces.

Def.: A sequence (x_n) in a metric space X is called a Cauchy sequence, iff

$$\forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n, m \geq N: d(x_n, x_m) < \varepsilon$$

Prop.: Every convergent sequence in a metric space

Prop.: Every convergent sequence in a metric space is also Cauchy. Ex. 4: Prove this!

Def.: A metric space X is called complete, iff every Cauchy sequence in X converges.
A complete normed space is called a Banach space.

Examples: $\mathbb{R}$, $\mathbb{C}$, $\mathbb{R}^n$, $\mathbb{C}^n$ are Banach spaces.
$\mathbb{Q}$ is not complete.