Def.: Let X, Y be topological spaces, $f: X \to Y$, and $a \in X$.

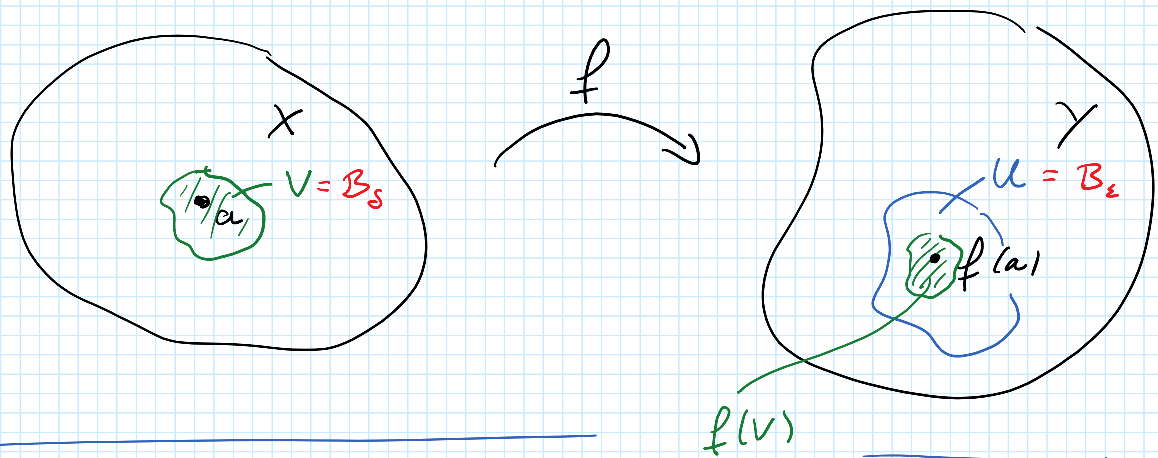
a) We say that f is <u>sequentially continuous</u> at a,

iff
$$\lim_{n\to\infty} x_n = a \Rightarrow \lim_{n\to\infty} f(x_n) = f(a).$$

b) We say that f is <u>continuous</u> at a,

iff $\forall\, U \in \mathcal{U}(f(a))\ \exists\, V \in \mathcal{U}(a) : f(V) \subset U.$

If a function is (sequ.) cont. at all points $a \in X$, then we say that f is (sequ.) cont. on X.



Prop.: If $f: X \to Y$ is continuous at $a \in X$, then f is also sequentially continuous at a.

Ex. 1: Prove this!

Sol. 1: Let $f: X \to Y$ is cont. at $a \in X$.

Let $\left(\lim_{n\to\infty} x_n = a\right)$. We need to show that $\lim_{n\to\infty} f(x_n) = f(a)$,

i.e. $\forall\, U \in \mathcal{U}(f(a))\ \exists\, N \in \mathbb{N} : \forall\, n \geq N \quad f(x_n) \in U.$

$\left(\ldots \underset{n\to\infty}{\ldots}\right) \quad \underset{n\to\infty}{\ldots}$

i.e. $\forall U \in \mathcal{U}(f(a))\ \exists N \in \mathbb{N} : \forall n \geq N \quad f(x_n) \in U$.

Let $U \in \mathcal{U}(f(a))$. Then by cont. of f at a there exists a ~~...~~ $V \in \mathcal{U}(a)$ s.t. $f(V) \subset U$. Since $\lim x_n = a$,

$\exists N \in \mathbb{N} : \forall n \geq N \quad x_n \in V \implies f(x_n) \in f(V) \subset U$. ☒

Prop.: „ε-δ-cont. in metric spaces"

A fct. $f : X \to Y$ between metric spaces X, Y is continuous at $a \in X$, iff

(i) $\forall \varepsilon > 0\ \exists \delta > 0 : f(B_\delta(a)) \subset B_\varepsilon(f(a))$.

(ii) $\forall U \in \mathcal{U}(f(a))\ \exists V \in \mathcal{U}(a) : f(V) \subset U$

Exerc. 2: Prove this!

Sol. 2: (i) $\Rightarrow$ (ii) Let $U \in \mathcal{U}(f(a))$. By def. of neighb. in metric spaces, $\exists \varepsilon > 0 \quad B_\varepsilon(f(a)) \subset U$.

$$\Rightarrow \exists \delta > 0 : f(\underbrace{B_\delta(a)}_{=:V}) \subset B_\varepsilon(f(a)) \subset U$$

(ii) $\Rightarrow$ (i) Let $\varepsilon > 0$. Then $B_\varepsilon(f(a)) \in \mathcal{U}(f(a))$

$\Rightarrow \exists V \in \mathcal{U}(a) : f(V) \subset U$. As before, $\exists \delta > 0$

$B_\delta(a) \subset V \implies f(B_\delta(a)) \subset f(V) \subset U = B_\varepsilon(f(a))$. ☒

Prop.: A fct. $f: X \to Y$ between metric spaces X, Y is cont. at $a \in X$, iff it is sequ. cont. at a.

Proof: „$\Leftarrow$": (by contraposition $(A \Rightarrow B) \Leftrightarrow (\neg B \Rightarrow \neg A)$)

Assume that f is not cont. at a, i.e.

$\exists\, \varepsilon > 0\ \forall\, \delta > 0 : f(B_\delta(a)) \not\subset B_\varepsilon(f(a))$.

For $\delta = \frac{1}{n}$ choose $x_n \in B_\delta(a) \setminus f^{-1}(B_\varepsilon(f(a))) \neq \emptyset$.

Then $\lim_{n \to \infty} x_n = a$ but $f(x_n) \notin B_\varepsilon(f(a))\ \forall n$.

$\Rightarrow f$ is not sequ. cont. $\blacksquare$

Thm.: Let X, Y be top. spaces. A map $f: X \to Y$ is continuous (on X), iff the preimage $f^{-1}(O) \subset X$ of any open set $O \subset Y$ is open.

cont. fct. that do not map open sets to open sets are e.g.

$\sin: \mathbb{R} \to \mathbb{R}$, $\sin(\mathbb{R}) = [-1, 1]$, $f: \mathbb{R} \to \mathbb{R}$, $x \mapsto f(x) \equiv a$.

$f(O) = \{a\}$

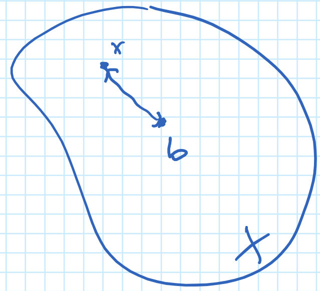
Examples: a) In a metric space (X, d) the distance fct. to a point $b \in X$,

$$\underset{\ddot{\mathrm{p}}}{d_b} : X \to [0, \infty),\ x \mapsto d_b(x) := d(x, b)$$

f is continuous. Ex. 3: Prove this!

Sol. 3: It suff. to show sequ. cont.

Let $a \in X$ and $\boxed{\lim_{n \to \infty} x_n = a}$



$$|f(x_n) - f(a)| = |d_b(x_n) - d_b(a)|$$

$$= |d(x_n, b) - d(a, b)|$$

$$\leq |d(x_n, a)| \xrightarrow{n \to \infty} 0$$

$$\Rightarrow \lim_{n \to \infty} d_b(x_n) = d_b(a). \quad \boxtimes$$

reversed Δ-inequ.

$$|d(x,y) - d(y,z)| \leq d(x,z)$$

Also $d : X \times X \to [0, \infty)$ is cont.

Remark: • Let (X, d_X) and (Y, d_Y) be metric spaces.

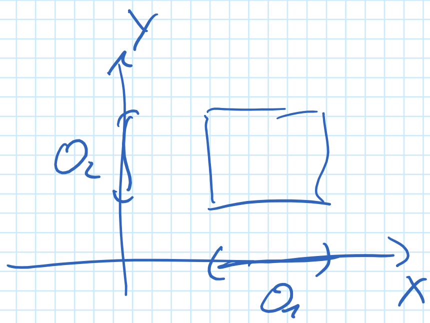
Then a metric on $X \times Y$ is for example

$$d((x_1, y_1), (x_2, y_2)) := \left(d_X(x_1, x_2)^p + d_Y(y_1, y_2)^p \right)^{1/p} \quad 1 \leq p < \infty.$$

• Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ top. spaces. Then the (product topology) on $X \times Y$ is generated by

$$\{ O_1 \times O_2 \mid O_1 \in \mathcal{T}_X, O_2 \in \mathcal{T}_Y \}$$

also called box topology.



• Let $(X_i, \mathcal{T}_i)$, $i \in I$, be top. spaces

Then the product top. on $\prod_{i \in I} X_i$

is generated by

$$\left\{ \prod_{i\in I} O_i \;\middle|\; O_i \in \mathcal{T}_i \text{ and } O_i \neq X_i \text{ only for finitely many } i \in I \right\}.$$

b) In a normed space $(V, \|\cdot\|)$ the norm

$$\|\cdot\| : V \to [0,\infty),$$

addition

$$+ : V \times V \to V, \quad (x,y) \mapsto x+y,$$

and multipl. by scalars

$$\cdot : \mathbb{K} \times V \to V, \quad (\lambda, v) \mapsto \lambda \cdot v$$

are all continuous.

c) The composition of cont. fct. is cont.:

If $f: X \to Y$ and $g: Y \to Z$ are cont., then also $g \circ f : X \to Z$ is cont.

Proof: Let $O \subset Z$ be open $\overset{g \text{ cont.}}{\Longrightarrow}$ $g^{-1}(O) \subset Y$ is open $\overset{f \text{ cont}}{\Longrightarrow}$ $\underbrace{f^{-1}(g^{-1}(O))}_{(g\circ f)^{-1}(O)} \subset X$ is open

$\Rightarrow$ $g \circ f$ is cont. ☒

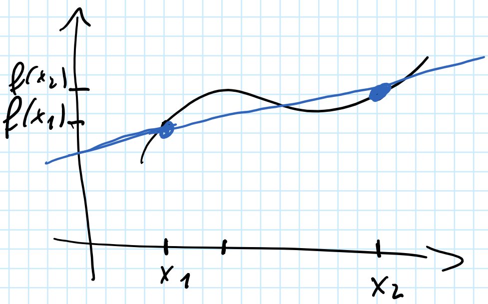
Def.: Let X, Y be metric spaces. A fct. $f: X \to Y$ is called Lipschitz-cont., iff there exist $L \geq 0$

is called <u>Lipschitz-cont.</u>, iff there exist $L \geq 0$ such that

$$\forall x_1, x_2 \in X : \boxed{d_Y(f(x_1), f(x_2)) \leq L \, d_X(x_1, x_2)}.$$

Then L is called a <u>Lipschitz constant</u> for f.

If f has a Lipschitz constant $L < 1$, then f is called a <u>contraction</u>.



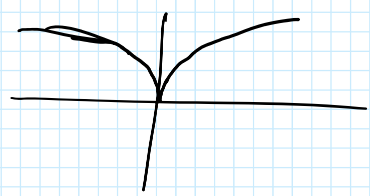
$$\boxed{\frac{d_Y(f(x_1), f(x_2))}{d_X(x_1, x_2)} \leq L}$$

$f(x) = ax + b$ is Lip. cont. with $L = a$

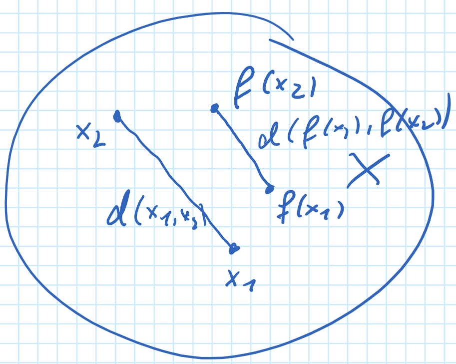
$f \in C^1(\mathbb{R})$ then $L = \sup_{x \in \mathbb{R}} |f'(x)|$

$f(x) = x^2$ is cont. but not Lipschitz cont.

$f(x) = \sqrt{|x|}$ —"—



<u>contraction:</u>



$f : X \to X$ contraction

Def.: a) Two top. spaces X, Y are homeomorphic, iff there exists a bicontinuous bijection $f: X \to Y$ (a homeomorphism)

b) A map $f: X \to Y$ between metric spaces is an isometry, iff

$$\forall x_1, x_2 \in X: d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2).$$

X and Y are isometric, iff there exists a bijective isometry $f: X \to Y$.

c) Two normed spaces V and W are isometrically isomorphic, iff there is a linear bijection (isomorphism) $A: V \to W$ such that

$$\forall v \in V: \|Av\|_W = \|v\|_V.$$

Example: The isometries of euclidian space $(\mathbb{R}^n, d_2)$ are translations, rotations and reflections, and compositions thereof (euclidian group).

Def.: Let X be a set, Y a metric space and $f_n: X \to Y$, $n \in \mathbb{N}$ and $f: X \to Y$ fcts.

(a) We say that f_n converges pointwise to f, iff

$$\forall x \in X: \lim d_Y(f_n(x), f(x)) = 0.$$

$$\forall x \in X: \underbrace{\lim_{n\to\infty} d_Y(f_n(x), f(x)) = 0}_{\Leftrightarrow \lim_{n\to\infty} f_n(x) = f(x)}.$$

(b) We say that f_n converges uniformly to f, iff

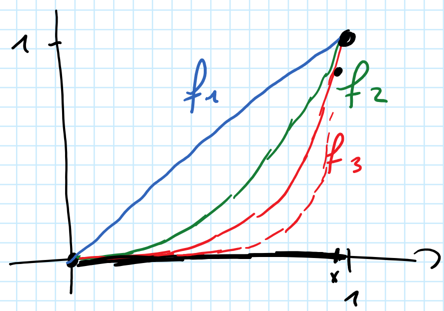
$$\lim_{n\to\infty} \sup_{x\in X} d_Y(f_n(x), f(x)) = 0$$

$$\sup_{x\in X} \|f_n(x) - f(x)\|_Y =: \|f_n - f\|_\infty$$

(b) $\Rightarrow$ (a) ; (a) $\not\Rightarrow$ (b)

Example: $f_n : [0,1] \to [0,1]$, $x \mapsto f_n(x) = x^n$

$$f_n(x) \xrightarrow{n\to\infty} \left\{ \begin{array}{ll} 0 & \text{for } x < 1 \\ 1 & \text{for } x = 1 \end{array} \right\} = f(x)$$



$\Rightarrow f_n \to f$ pointwise, but not uniformly: $\sup_{x\in[0,1]} |f_n(x) - f(x)| = 1$

$[\lim_{n\to\infty} f_n(1) = \lim_{n\to\infty} 1 = 1]$ $\boxed{\lim 1^n = \lim 1 = 1}$

$\lim \underbrace{(-1)^n}_{(-1,\,1,\,-1,\,1,\,-1,\,\dots)}$

If $(Y, \|\cdot\|)$ is a normed space, then $f_n \to f$ uniformly, iff $\lim_{n\to\infty} \|f_n - f\|_\infty = 0$.

Thm.: Uniform limits of cont. fcts are cont.

Thm.: <u>Uniform limits of cont. fcts are cont.</u>

Let $f_n : X \to Y$ be a <u>sequence of cont.</u> fcts. (X top. and Y metric space) and let $f_n \to f$ <u>uniformly</u>. Then f is continuous.

Corollary: Let X be a top. space, $(Y, \|\cdot\|_Y)$ a complete normed space and $C_b(X,Y)$ the space of cont. bounded fcts equipped with the $\|\cdot\|_\infty$-norm.

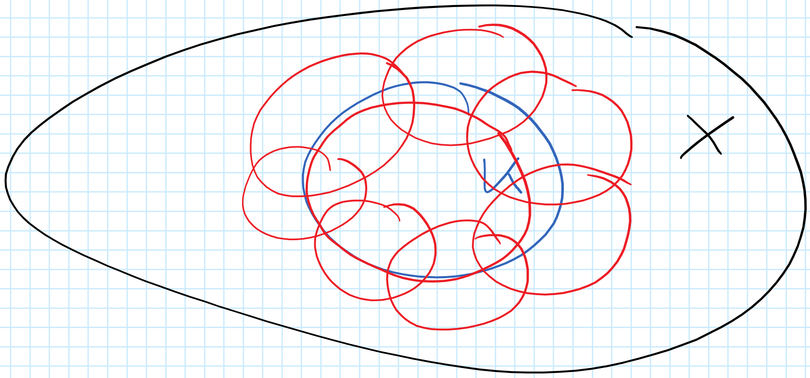
$$\|f\|_\infty := \sup_{x \in X} \|f(x)\|_Y .$$

Then the normed space $(C_b(X,Y), \|\cdot\|_\infty)$ is complete.

Def: Let $(X, \mathcal{T})$ be a topological space and $Y \subset X$. A family $(U_i)_{i \in I}$ of open sets, $U_i \in \mathcal{T}\ \forall i \in I$, is called an <u>open cover</u> of Y, iff

$$Y \subset \bigcup_{i \in I} U_i$$

A set $K \subset X$ is called <u>compact</u>, iff any open cover $(U_i)_{i \in I}$ of K admits a finite subcover, i.e. there exist $i_1, \dots, i_n \in I$ such that $K = U_{i_1} \cup U_{i_2} \cup \dots \cup U_{i_n}$



Examples: (a) Every finite subset $K = \{x_1, \dots, x_n\}$ of a top. space is compact.

(b) $(0,1] \subset \mathbb{R}$ is not compact: The open cover

$$\boxed{(0,1] \subset \bigcup_{n=2}^{\infty} \left(\tfrac{1}{n}, 2\right)}$$ admits no finite subcover.

Thm.: Bolzano-Weierstraß

Let $K \subset X$ be compact. Then any sequence in K has a cluster point in K.

Ex. 4: Prove this! | $x_n = n$ has no cluster point

Sol. 4: Indirect proof: Let $K \subset X$ be compact and (x_n) a sequence in K, i.e. $x_n \in K \ \forall n \in \mathbb{N}$.

Assume that (x_n) has no cluster point in K.

$\Rightarrow \forall x \in K \ \exists \underset{\text{open}}{U_x} \in \mathcal{U}(x)$ s.t. $|\{n \in \mathbb{N} \mid x_n \in U_x\}| < \infty$

Since $K = \bigcup_{x \in K} U_x$, $\bigcup_{x \in K} U_x$ forms an open cover of K.

$\overset{K \text{ compact}}{\Longrightarrow} \ \exists \{a_1, \dots, a_m\} \subset K : \ K \subset \bigcup^{m} U_{a_i}$

$\overset{K \text{ compact}}{\Longrightarrow} \quad \exists \{a_1, \dots, a_m\} \subset K \; : \; K \subset \bigcup_{j=1}^{m} U_{a_j}$

$$\underbrace{\left| \{ n \in \mathbb{N} \mid x_n \in K \} \right|}_{= |\mathbb{N}| = \infty} \leq \left| \{ n \in \mathbb{N} \mid x_n \in \bigcup_{j=1}^{m} U_{a_j} \} \right| < \infty .$$

$\lightning$

A a set then $|A| = \#$ elements of A if A is finite
otherwise $|A| = \infty$

Rem.: In metric spaces also the converse is true.

Prop.: Let $f: X \to Y$ be continuous and $K \subset X$ compact set. Then $f(K) \subset Y$ is compact.

Prop.: Let X be a top. space and $K \subset X$ compact. Then any closed subset $A \subset K$ is also compact. If X is Hausdorff space, then K is closed.

Def.: Let X be a metric space.

(a) A subset $B \subset X$ is bounded, iff

$$\exists C \in \mathbb{R} \; \forall x, y \in B : d(x,y) \leq C.$$

(b) The diameter of a set $Y \subset X$ is

(b) The <u>diameter</u> of a set $Y \subset X$ is

$$\operatorname{diam}(Y) := \sup\{ d(x,y) \mid x, y \in Y \} \in [0,\infty) \cup \{\infty\}$$

Thm.: Let X be a metric space and $K \subset X$ compact.
Then K is bounded and closed.

Thm. Heine-Borel

A subset K of a finite dimensional normed space is compact, iff it is bounded and closed.

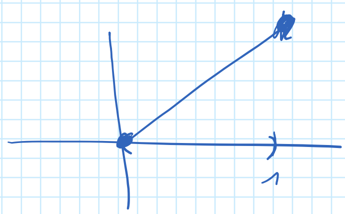
Thm.: Weierstraß

Let $f: K \to \mathbb{R}$ be continuous and K compact.
Then f is bounded ($f(K) \subset \mathbb{R}$ is bounded)
and attains its ~~maximum~~ supremum and its ~~minimum~~ infimum.
$\Leftrightarrow$ it has a max. and a min

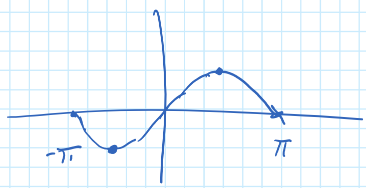
$f: [0,1] \to \mathbb{R}$, $f(x) = x$, $\sup(\underbrace{f((0,1))}_{=(0,1)}) = 1$

not compact

but $\nexists x \in (0,1) : f(x) = 1$



$\sin : (-\pi, \pi) \to \mathbb{R}$



Def.: Let X, Y be metric spaces and

Def.: Let X, Y be metric spaces and $A \subset C(X,Y)$. The set A is called equicontinuous at $x \in X$, iff

$\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall f \in A: \; f(B_\delta(x)) \subset B_\varepsilon(f(x))$.

Thm.: Arzela-Ascoli

Let X be a compact metric space and consider $C(X, \mathbb{C})$ equipped with the $\|\cdot\|_\infty$-norm.

A subset $K \subset C(X, \mathbb{C})$ is compact, iff it is closed, bounded pointwise (i.e. $\forall x \in X$: $\sup_{f \in K} |f(x)| < \infty$) and equicontinuous.

Def.: Let X be a top. space.

If X is the union of two disjoint, open, non-empty sets, then X is disconnected, otherwise connected.

X is path-connected, if any two points $x_0, x_1 \in X$ can be connected by a continuous path, i.e. there exist $\gamma: [0,1] \to X$ cont. with $\gamma(0) = x_0$ and $\gamma(1) = x_1$.

Prop.: X path-connected $\overset{\not\Leftarrow}{\Rightarrow}$ X connected

Prop.: Let O be an open subset of a normed space. Then O is connected iff it is path-connected.

Prop.: Let $f: X \to Y$ be continuous and $A \subset X$ (path) connected. Then also $f(A) \subset Y$ is (path) connected.

Def.: A non-empty, open, connected subset $D \subset X$ of a top. space X is called a domain.

Def.: $f: X \to Y$ metric space is called bounded, iff $f(X) \subset Y$ is bounded.

$f: \mathbb{R} \to \mathbb{R}$, $f(x) = x$

Def: Bounded linear maps and their norms

A linear map $A: V \to W$ between normed spaces is called bounded, iff $A(B_1(0))$ is bounded, i.e. $\exists C \in \mathbb{R} \ \forall x \in V$

$$\|Ax\|_W \leq C \|x\|_V .$$

The smallest such constant C is called the operator norm of A, i.e.

$$\|A\|_{op} := \sup\{ \|Ax\|_W \mid x \in \overline{B_1(0)} \}$$

The space of bounded lin. maps $V \to W$ is denoted by $\mathcal{L}(V,W)$ or $\mathcal{B}(V,W)$ and $\|\cdot\|_{op}$ is norm on $\mathcal{L}(V,W)$.

Note that for $A \in \mathcal{L}(V,W)$ we have for all $x \in V$

$$\boxed{\|Ax\|_W \leq \|A\|_{op} \cdot \|x\|_V}$$

Cauchy-Schwarz

$$|\langle x, y \rangle| \leq \|x\| \cdot \|y\|$$

If $(W, \|\cdot\|_W)$ is a Banach space, then also $(\mathcal{L}(V,W), \|\cdot\|_{op})$ is also complete.

If $\dim V < \infty$, then all lin. maps $V \to W$ are bounded

$$\begin{pmatrix} 1 & & & & 0 \\ & 2 & & & \\ & & 3 & & \\ & & & 4 & \\ 0 & & & & \ddots \end{pmatrix}$$

0