Def.: Let X, Y be topological spaces, $f: X \to Y$, and $a \in X$.

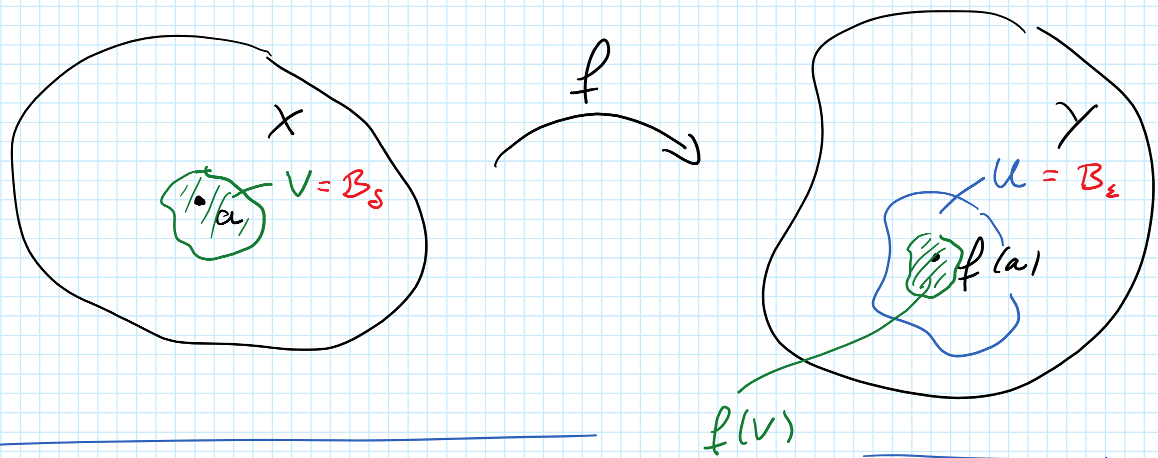
a) We say that f is *sequentially continuous* at a,

iff $$\lim_{n\to\infty} x_n = a \implies \lim_{n\to\infty} f(x_n) = f(a).$$

b) We say that f is *continuous* at a,

iff $$\forall U \in \mathcal{U}(f(a)) \ \exists V \in \mathcal{U}(a) : f(V) \subset U.$$

If a function is (sequ.) cont. at all points $a \in X$, then we say that f is *(sequ.) cont. on X*.



Prop.: If $f: X \to Y$ is continuous at $a \in X$, then f is also sequentially continuous at a.

Ex. 1: Prove this!

Prop.: „ε-δ-cont. in metric spaces"

A fct. $f: X \to Y$ between metric spaces X, Y

A fct. $f: X \to Y$ between metric spaces X, Y is continuous at $a \in X$, iff

$$\forall \varepsilon > 0 \; \exists \delta > 0 : f(B_\delta(a)) \subset B_\varepsilon(f(a)).$$

Exerc. 2: Prove this!

Prop.: A fct. $f: X \to Y$ between metric spaces X, Y is cont. at $a \in X$, iff it is sequ. cont. at a.

Proof: „$\Leftarrow$": (by contraposition $(A \Rightarrow B) \Leftrightarrow (\neg B \Rightarrow \neg A)$)

Assume that f is not cont. at a, i.e.

$$\exists \varepsilon > 0 \; \forall \delta > 0 : f(B_\delta(a)) \not\subset B_\varepsilon(f(a)).$$

For $\delta = \frac{1}{n}$ choose $x_n \in B_\delta(a) \setminus f^{-1}(B_\varepsilon(f(a))) \neq \emptyset$.

Then $\lim_{n \to \infty} x_n = a$ but $f(x_n) \notin B_\varepsilon(f(a)) \; \forall n$.

$\Rightarrow f$ is not sequ. cont. $\blacksquare$

Thm.: Let X, Y be top. spaces. A map $f: X \to Y$ is continuous (on X), iff the preimage $f^{-1}(O) \subset X$ of any open set $O \subset Y$ is open.

Examples: a) In a metric space (X, d) the distance fct. to a point $b \in X$,

$$d_b : X \to [0,\infty), \quad x \mapsto d_b(x) := d(x,b)$$

is continuous. Ex. 3: Prove this!

Also $d : X \times X \to [0,\infty)$ is cont.

b) In a normed space $(V, \|\cdot\|)$ the norm

$$\|\cdot\| : V \to [0,\infty),$$

addition

$$+ : V \times V \to V, \quad (x,y) \mapsto x+y,$$

and multipl. by scalars

$$\cdot : \mathbb{K} \times V \to V, \quad (\lambda, v) \mapsto \lambda \cdot v$$

are all continuous.

c) The composition of cont. fct. is cont.:

If $f : X \to Y$ and $g : Y \to Z$ are cont., then also $g \circ f : X \to Z$ is cont.

Proof: Let $O \subset Z$ be open $\overset{g \text{ cont.}}{\Longrightarrow} g^{-1}(O) \subset Y$ is open $\overset{f \text{ cont}}{\Longrightarrow} \underbrace{f^{-1}(g^{-1}(O))}_{(g\circ f)^{-1}(O)} \subset X$ is open

$\Rightarrow g \circ f$ is cont. $\square$

Def.: Let X, Y be metric spaces. A fct. $f : X \to Y$ is called Lipschitz-cont., iff there exist $L \geq 0$ such that such that

$$\forall x_1, x_2 \in X : d_Y(f(x_1), f(x_2)) \leq L\, d_X(x_1, x_2).$$

Then L is called a Lipschitz constant for f.
If f has a Lipschitz constant $L < 1$, then f is called a contraction.

Def.: a) Two top. spaces X, Y are homeomorphic, iff there exists a bicontinuous bijection $f : X \to Y$ (a homeomorphism)

b) A map $f : X \to Y$ between metric spaces is an isometry, iff

$$\forall x_1, x_2 \in X : d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2).$$

X and Y are isometric, iff there exists a bijective isometry $f : X \to Y$.

c) Two normed spaces V and W are isometrically isomorphic, iff there is a linear bijection (isomorphism) $A : V \to W$ such that

$$\forall v \in V : \|Av\|_W = \|v\|_V .$$

Example: The isometries of euclidian space $(\mathbb{R}^n, d_2)$ are translations, rotations and reflections, and compositions thereof (euclidian group).

Def.: Let X be a set, Y a metric space and $f_n : X \to Y$, $n \in \mathbb{N}$ and $f : X \to Y$ fcts.

(a) We say that f_n converges pointwise to f, iff

$$\forall x \in X : \underbrace{\lim_{n\to\infty} d_Y(f_n(x), f(x)) = 0}_{\Leftrightarrow \lim_{n\to\infty} f_n(x) = f(x)}.$$

(b) We say that f_n converges uniformly to f, iff

$$\lim_{n\to\infty} \underbrace{\sup_{x\in X} d_Y(f_n(x), f(x))}_{\sup_{x\in X} \|f_n(x) - f(x)\|_Y =: \|f_n - f\|_\infty} = 0$$

If $(Y, \|\cdot\|)$ is a normed space, then $f_n \to f$ uniformly, iff $\lim_{n\to\infty} \|f_n - f\|_\infty = 0$.

Thm.: Uniform limits of cont. fcts are cont.
Let $f_n : X \to Y$ be a sequence of cont. fcts. (X top. and Y metric space) and let $f_n \to f$ uniformly. Then f is continuous.

Corollary: Let X be a top. space, $(Y, \|\cdot\|_Y)$ a complete normed space and $C_b(X, Y)$ the space of cont. bounded fcts equipped with the $\|\cdot\|_\infty$-norm.

of cont. bounded fcts equipped with the $\|\cdot\|_\infty$-norm.

$$\|f\|_\infty := \sup_{x \in X} \|f(x)\|_Y .$$

Then the normed space $(C_b(X,Y), \|\cdot\|_\infty)$ is complete.

<u>Def:</u> Let $(X, \mathcal{T})$ be a topological space and $Y \subset X$. A family $(U_i)_{i \in I}$ of open sets, $U_i \in \mathcal{T}$ $\forall i \in I$, is called an <u>open cover</u> of Y, iff

$$Y \subset \bigcup_{i \in I} U_i$$

A set $K \subset X$ is called <u>compact</u>, iff any open cover $(U_i)_{i \in I}$ of K admits a finite subcover, i.e. there exist $i_1, \dots, i_n \in I$ such that $K = U_{i_1} \cup U_{i_2} \cup \dots \cup U_{i_n}$

<u>Examples:</u> (a) Every finite subset $K = \{x_1, \dots, x_n\}$ of a top. space is compact.

(b) $\underline{(0,1)} \subset \mathbb{R}$ is not compact: The open cover

$$(0,1) = \bigcup_{n=2}^{\infty} \left(\tfrac{1}{n}, 1\right)$$ admits no finite subcover.

Thm.: Bolzano-Weierstraß

Let $K \subset X$ be compact. Then any sequence in K has a cluster point in K.

Ex. 4: Prove this!

Rem.: In metric spaces also the converse is true.

Prop.: Let $f : X \to Y$ be continuous and $K \subset X$ compact set. Then $f(K) \subset Y$ is compact.

Prop.: Let X be a top. space and $K \subset X$ compact. Then any closed subset $A \subset K$ is also compact. If X is Hausdorff space, then K is closed.

Def.: Let X be a metric space.

(a) A subset $B \subset X$ is bounded, iff

$$\exists\, C \in \mathbb{R}\ \forall x, y \in B : d(x,y) \leq C.$$

(b) The diameter of a set $Y \subset X$ is

$$\operatorname{diam}(Y) := \sup \{ d(x,y) \mid x, y \in Y \} \in [0,\infty) \cup \{\infty\}$$

Thm.: Let X be a metric space and $K \subset X$ compact.

Thm.: Let X be a metric space and $K \subset X$ compact.
Then K is bounded and closed.

Thm. Heine-Borel
A subset K of a finite dimensional normed space is compact, iff it is bounded and closed.

Thm.: Weierstraß

Let $f: K \to \mathbb{R}$ be continuous and K compact.
Then f is bounded ($f(K) \subset \mathbb{R}$ is bounded) and attains its maximum and its minimum.

Def.: Let X, Y be metric spaces and $A \subset C(X, Y)$. The set A is called <u>equicontinuous</u> at $x \in X$, iff

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall f \in A: \ f(B_\delta(x)) \subset B_\varepsilon(f(x)).$$

Thm.: Arzela-Ascoli

Let X be a compact metric space and consider $C(X, \mathbb{C})$ equipped with the $\|.\|_\infty$-norm.
A subset $K \subset C(X, \mathbb{C})$ is compact, iff

A subset $K \subset C(X, \mathbb{C})$ is compact, iff it is closed, bounded pointwise (i.e. $\forall x \in X$: $\sup_{f \in K} |f(x)| < \infty$) and equicontinuous.

Def.: Let X be a top. space.
If X is the union of two disjoint, open, non-empty sets, then X is disconnected, otherwise connected.
X is path-connected, if any two points $x_0, x_1 \in X$ can be connected by a continuous path, i.e. there exist $\gamma: [0,1] \to X$ cont. with $\gamma(0) = x_0$ and $\gamma(1) = x_1$.

Prop.: X path-connected $\overset{\not\Leftarrow}{\Rightarrow}$ X connected

Prop.: Let O be an open subset of a normed space. Then O is connected iff it is path-connected.

Prop.: Let $f: X \to Y$ be continuous and $A \subset X$ (path) connected. Then also $f(A) \subset Y$ is (path) connected.

Def.: A non-empty, open, connected subset

<u>Def.</u>: A non-empty, open, connected subset $D \subset X$ of a top. space X is called a <u>domain</u>.