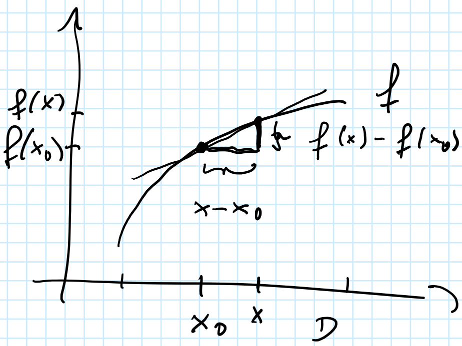
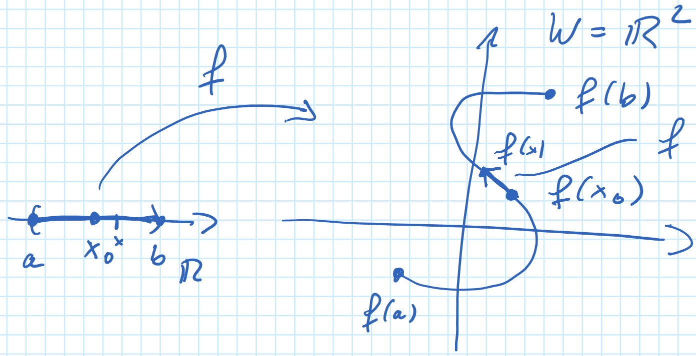
Recall that for $f: \mathbb{R} \supset D \longrightarrow \cancel{\mathbb{R}}\ (W, \|\cdot\|)$

$$\underline{f'(x_0)} := \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

[$\forall (x_n)$ in $D \setminus \{x_0\}$ with $\lim_{n\to\infty} x_n = x_0$ $\lim_{n\to\infty} \frac{f(x_n) - f(x_0)}{x_n - x_0}$ exists.]





$f'(x_0)$ = slope of tangent to the graph at $(x_0, f(x_0))$

$f'(x_0) \in W$ = velocity vector

<u>Def.:</u> Let $n \in \mathbb{N}$, $D \subset \mathbb{R}^n$ open, $(W, \|\cdot\|)$ a normed space. For $x \in D$ and $j \in \{1, .., n\}$ a function $f: D \to W$ is called <u>partially differentiable</u> in the jth coord. direction at x, iff the limit

$$\lim_{h \to 0} \frac{f(x + h e_j) - f(x)}{h} = \lim_{h \to 0} \frac{f(x_1, x_2, .., x_j + h, .., x_n) - f(x_1, \dots, x_n)}{h}$$

exists. One writes

$$\frac{\partial f}{\partial x_j}(x) = \partial_j f(x) := \lim_{h \to 0} \frac{f(x + h e_j) - f(x)}{h}$$

and calls the vector $\partial_j f(x) \in W$ the jth <u>partial</u>

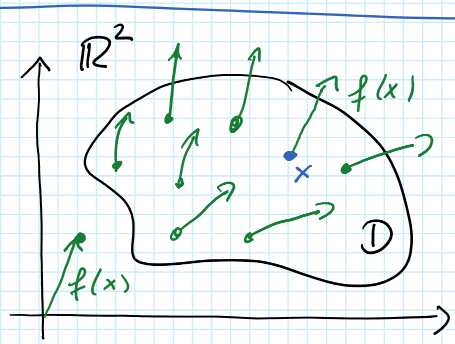
and calls the vector $\partial_j f(x) \in W$ the jth partial derivative at x.

If f is partially diff. and the partial derivatives $\partial_j f : D \to W$ are cont. fcts., then f is called cont. part. diff. The vector space of c.p.d. fcts on $D \subset \mathbb{R}^n$ is denoted by $C^1(D, W)$.

The gradient of f at x is

$$\nabla f(x) := (\partial_1 f(x), \dots, \partial_n f(x)) \in W^n$$

Def.: A map $f : \mathbb{R}^n \supset D \to \mathbb{R}^n$ is called a vector field.



Ex.: The gradient
$\nabla f : D \to \mathbb{R}^n$
of a fct. $f : D \to \mathbb{R}$, $D \subset \mathbb{R}^n$
defines a vector field.

Def.: A fct. $f : \mathbb{R}^n \supset D \to W$ ($\in \mathbb{K}$-VSp.) is called r-times cont. part. diff., iff for all $j = (j_1, \dots, j_r)$, $j_i \in \{1, \dots, n\}$

- f is c.p.d.
- $\partial_{j_1} f$ is c.p.d.
- $\partial_{j_2} \partial_{j_1} f$ is c.p.d.

$\vdots$

- $\partial_{j_1} f$ is c.p.d.
- $\partial_{j_2}\partial_{j_1} f$ is c.p.d.

$\vdots$

- $\partial_{j_r} \cdots \partial_{j_1} f$ is continuous

The $\mathbb{K}$-vector space of r-times c.p.d. fcts is denoted by $\boxed{C^r(D,W).}$

Def.: Let $D \subset \mathbb{R}^n$, $g \in C^1(D,\mathbb{R}^n)$, $f \in C^2(D,\cancel{\mathbb{R}}\,W)$.

Then
$$\operatorname{div} g : D \to \mathbb{R},\ x \mapsto \operatorname{div} g(x) := \underline{\sum_{j=1}^{n} \frac{\partial g_j}{\partial x_j}(x)}$$

is called the <u>divergence</u> of g,

$$\operatorname{curl} g : D \to \mathbb{R}^3,\ x \mapsto \operatorname{curl} g(x) := \begin{pmatrix} \partial_2 g_3(x) - \partial_3 g_2(x) \\ \partial_3 g_1(x) - \partial_1 g_3(x) \\ \partial_1 g_2(x) - \partial_2 g_1(x) \end{pmatrix}$$

for $n=3$ is called the <u>curl</u> of g,

and

$$\Delta f : D \to \cancel{\mathbb{R}}\,W,\ x \mapsto \Delta f(x) := (\operatorname{div}(\nabla f)(x)) = \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_j^2}(x)$$

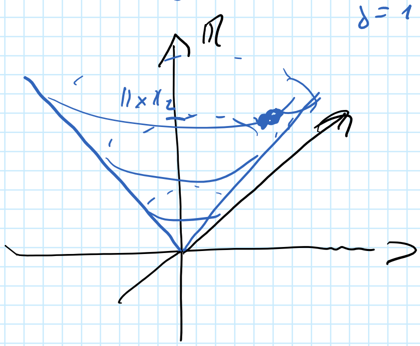
is called <u>Laplace</u> of f.

<u>Ex. 1:</u> Let $g : \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto g(x) = x$ $\quad (g = id)$

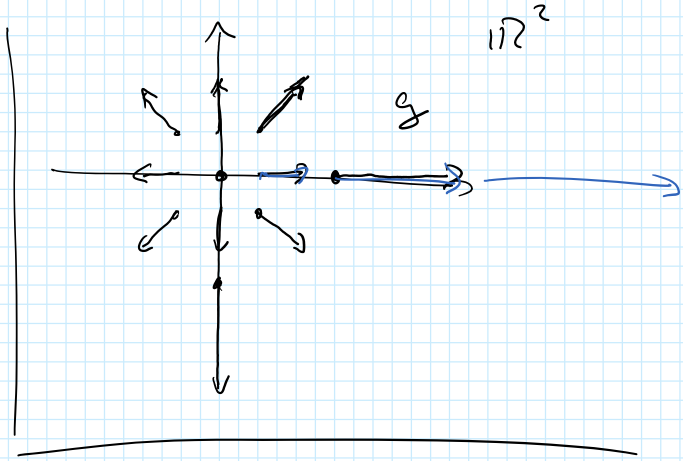
and $f : \underline{\mathbb{R}^n \setminus \{0\}} \to \mathbb{R}$, $x \mapsto \|x\|_2$. $\quad (n=1)\ f(x) = |x|$

Compute $\operatorname{div}(g)$ and Δf.

Sol. 1: $\operatorname{div}(g)(x) = \sum_{j=1}^{n} \frac{\partial g_j(x)}{\partial x_j} = \sum_{j=1}^{n} \frac{\partial x_j}{\partial x_j} = \sum_{j=1}^{n} 1 = n.$



$\mathbb{R}^2$



$$\Delta f(x) = \operatorname{div}(\nabla f)(x)$$

$$(\nabla f(x))_j = \partial_j f(x) = \partial_j \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$$

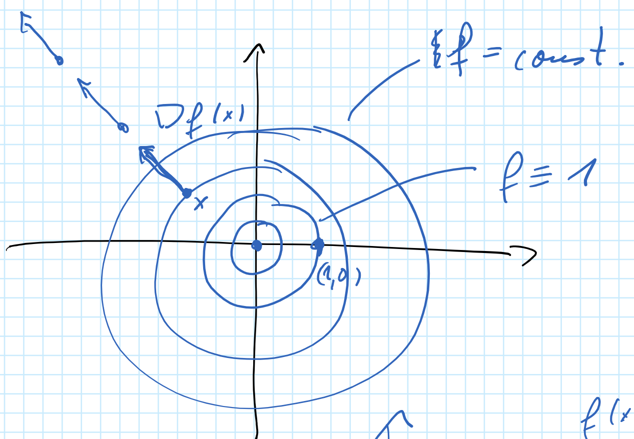
$$= \frac{1}{2} \frac{2x_j}{\sqrt{x_1^2 + \dots + x_n^2}} = \frac{x_j}{\|x\|_2} \Rightarrow \nabla f(x) = \frac{x}{\|x\|_2}$$

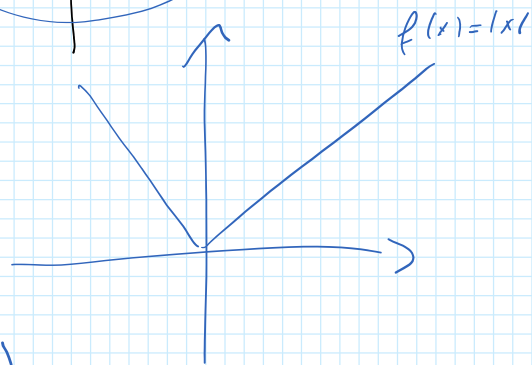
$$\Delta f(x) = \operatorname{div}\left(\frac{x}{\|x\|_2}\right) =$$

$$= \sum_{j=1}^{n} \partial_j \frac{x_j}{\|x\|_2} =$$

$$= \sum_{j=1}^{n} \left(\frac{1}{\|x\|_2} - \frac{1}{\cancel{2}} \frac{x_j \cdot \cancel{2} x_j}{\|x\|_2^3} \right)$$

$$= \frac{n}{\|x\|_2} - \frac{\|x\|_2^2}{\|x\|_2^3} = \frac{n-1}{\|x\|_2}.$$





Thm. (Schwarz)

Let $f \in C^2(D, W)$, $D \subset \mathbb{R}^n$.

Then $\forall x \in D$, $j, i \in \{1, \dots, n\}$

$$\partial_i \partial_j f(x) = \partial_j \partial_i f(x)$$

$$\partial_j \partial_i f(x) = \partial_i \partial_j f(x)$$

Corollary: Let $D \subset \mathbb{R}^3$, $f \in C^2(D,\mathbb{R})$ and $g \in C^2(D,\mathbb{R}^3)$. Then $\text{curl}(\nabla f) = 0$ and $\text{div}(\text{curl}\, g) = 0$

Def.: Let V be a real normed space, W a normed space, $D \subset V$ open and $f: D \to W$.
The directional derivative of f at $x \in D$ in the direction $v \in V$ is

$$\partial_v f(x) := \lim_{h \to 0} \frac{f(x+hv) - f(x)}{h} = \frac{d}{dh} f(x+hv)\Big|_{h=0}$$

if the limit exists.

Example: For $f: \mathbb{R}^n \to W$ we have $\partial_{e_j} f(x) = \partial_j f(x)$.

Different viewpoint: Derivative as lin. approx.

For $f: \mathbb{R} \to W$ differentiability at $x_0 \in \mathbb{R}$ means

$$0 = \lim_{x \to x_0} \left(\frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \right) =: \lim_{x \to x_0} \frac{\varphi(x, x_0)}{x - x_0}$$

where $\varphi(x, x_0) = f(x) - f(x_0) - f'(x_0)(x - x_0)$

or, after reshuffling

or, after reshuffling

$$\boxed{f(x) = \underbrace{f(x_0) + f'(x_0)\cdot(x-x_0)}_{(*)} + \varphi(x,x_0)}$$

where

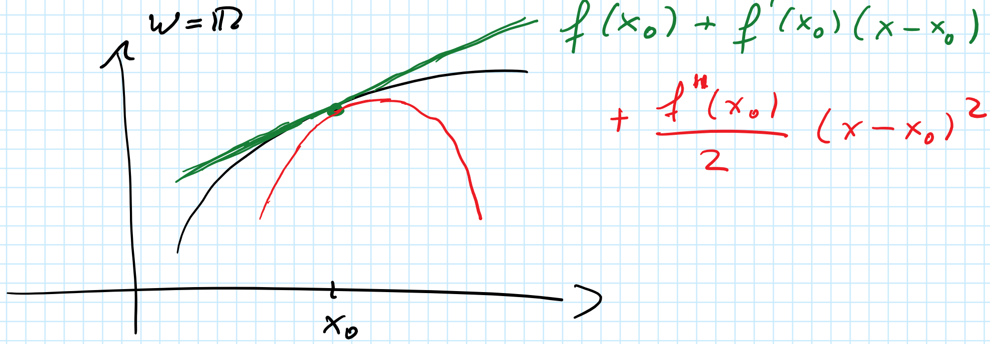
$$\varphi(x,x_0) = o(|x-x_0|) :\Longleftrightarrow \lim_{x\to x_0} \frac{|\varphi(x,x_0)|}{|x-x_0|} = 0$$

The map $\mathbb{R} \to W$, $x \mapsto f'(x_0)\cdot x$ is $\mathbb{R}$-linear

and the map $\mathbb{R} \to W$, $x \mapsto f(x_0) + f'(x_0)(x-x_0)$

is affine-$\mathbb{R}$-linear.

Hence, we think of $(*)$ as the (affine) linear approximation to f near x_0.



Def.: Total derivative

Let V be a finite dim. real vector space, W a normed space, $G \subset V$ open and $f: G \to W$.

f is differentiable at $x_0 \in G$, iff there exist an $\mathbb{R}$-linear map $A: V \to W$ such that

$$\boxed{\lim_{x\to x_0} \frac{f(x) - (f(x_0) + A(x-x_0))}{\|x-x_0\|_V} = 0}$$

(here $\|\cdot\|_V$ is any norm on V)

Then A is uniquely determined (Ex. 2: prove this!), is denote by $Df|_x$, and called the total derivative is denote by $Df|_{x_0}$, and called the total derivative or the differential of f at x_0.

If $f: G \to W$ is diff. at all $x \in G$, then f is called differentiable on G and

$$\boxed{Df : G \to \mathcal{L}(V, W), \; x \mapsto Df|_x}$$

is a fct. on G taking values in the linear maps from V to W.

Sol. 2: Let $\hat{A}: V \to W$ be linear and also satisfy

$$\lim_{x \to x_0} \frac{f(x) - f(x_0) - \tilde{A}(x - x_0)}{\|x - x_0\|_V} = 0$$

$$\Rightarrow \underset{0 =}{} \lim_{x \to x_0} \frac{A(x - x_0) - \tilde{A}(x - x_0)}{\|x - x_0\|_V} = \lim_{x \to x_0} \frac{\overbrace{(A - \tilde{A})}^{B} \overbrace{(x - x_0)}^{=h}}{\|x - x_0\|_V}$$

i.e. $\lim_{n \to \infty} \frac{B h_n}{\|h_n\|_V} = 0 \quad \forall (h_n)$ with $\lim_{n \to \infty} h_n = 0$

$v \neq 0$ Let (λ_n) in $\mathbb{R}$ with $\lambda_n \xrightarrow{n \to \infty} 0$

$$B v = \|v\| \cdot \frac{B v}{\|v\|}$$

$$= \|v\| \frac{B(\lambda_n v)}{\|\lambda_n v\|} \xrightarrow{n \to \infty} 0 \quad \Rightarrow B v = 0.$$

lin. of $B = A - \tilde{A}$

Rem.: $f: G \to W$ is diff. at x_0

$$\Rightarrow \quad f(x) = f(x_0) + Df|_{x_0}(x - x_0) + o(\|x - x_0\|_V)$$

$$\Rightarrow \boxed{f(x) = f(x_0) + Df|_{x_0} \underbrace{(x-x_0)}_{\to 0} + \underbrace{o(\|x-x_0\|_V)}_{\to 0 \text{ faster!}}}$$

Example: Let $L: V \to W$ be an $\mathbb{R}$-lin. map.

Then $L(x) = L(x_0 + x - x_0) = L(x_0) + L(x-x_0)$

$$=: L(x_0) + DL|_{x_0}(x-x_0)$$

and hence $DL|_{x_0} = L$. $\quad$ $f(x) = x \;,\; f'(x) = 1 \quad ?$

Thm.: Let $G \subset \mathbb{R}^n$ and let $f: G \to \mathbb{K}^m$ be differentiable at $x_0 \in G$. Then

$$(Df|_{x_0})_{ij} = \frac{\partial f_i}{\partial x_j}(x_0)$$

or, more explicitly,

$$Df|_{x_0} = \underbrace{\begin{pmatrix} \partial_1 f_1(x_0) & \cdots & \partial_n f_1(x_0) \\ \vdots & & \\ \partial_1 f_m(x_0) & \cdots & \partial_n f_m(x_0) \end{pmatrix}}_{n \text{ columns}} \Bigg\} m \text{ rows}$$

$$= \begin{pmatrix} \nabla f_1(x_0) \\ \vdots \\ \nabla f_m(x_0) \end{pmatrix} \quad \text{„Jacobi matrix"}$$

Thm.: Let $G \subset \mathbb{R}^n$ open and $f \in C^1(G, \mathbb{K}^m)$. Then f is differentiable.

cont. part. diff.

Then f is differentiable. cont. part. diff.

cont. part. diff. $\Rightarrow$ differentiable $\Rightarrow$ part. diff.

$\Downarrow$

continuous

None of the implications holds in the reversed direct.!

But

cont. part. diff $\Leftrightarrow$ diff. with cont. derivative

Thm.: Chain rule

Let U, V be finite dim. real vector spaces, W a normed space, $G \subset U$, $H \subset V$ open and $g: G \to V$, $f: H \to W$ maps with $g(G) \subset H$, i.e.

$$U \supset G \xrightarrow{g} H \subset V \xrightarrow{f} W.$$

If g is diff. at $x \in G$ and f is diff. at $g(x) \in H$, then $f \circ g: G \to W$ is diff. at x and

$$\boxed{D(f \circ g)|_x = \underbrace{Df|_{g(x)}}_{W \leftarrow V} \cdot \underbrace{Dg|_x}_{V \leftarrow U}}$$

Corollary: For $f \in C^1(G, W)$, $G \subset V$, $x_0 \in G$, and $v \in V$ we have

$$\partial_v f(x_0) = Df|_{x_0} v$$

$$\partial_v f(x_0) = Df|_{x_0} v$$

Ex. 3: prove this!

$g: \mathbb{R} \supset I \to G,\ h \mapsto x_0 + hv$

Sol. 3: $\partial_v f(x_0) = \frac{d}{dh} f(\underbrace{x_0 + hv}_{=: g(h)})\Big|_{h=0} = \frac{d}{dh}(f \circ g)(h)\Big|_{h=0}$

$= Df|_{g(0)}\, g'(0) = Df|_{x_0} v$

For $f: \overset{V}{\overset{\cup}{G}} \to W$ the differential Df is a map

$\boxed{Df: G \to \mathcal{L}(V, W).}$ Thus the „second derivative"

$D(Df)$ is a map $\quad g(v_1)(v_2) \in W \qquad g_{ij}^{l} \ldots$

$\qquad\qquad\qquad\qquad \cap \qquad \cong g(v_1, v_2)$

$$D(Df): G \to \mathcal{L}(V, \mathcal{L}(V, W)) \cong \mathcal{L}_2(V \times V, W)$$

$:=$ bilinear maps $V \times V \to W$

and the rth derivative

$$D^r f: G \to \mathcal{L}_r(\underbrace{V \times \cdots \times V}_{r\text{-copies}}, W).$$

$g(v_1, v_2) = (g_{ij}^{l} v_1^i v_2^j)\, w_l$

Thm. Taylor

Let $G \subset V$ open, $x_0 \in G$, and $\delta > 0$ s.t. $B_\delta(x_0) \subset G$.

Then for any $f \in C^r(G, W)$ and $x \in B_\delta(x_0)$

$f(x) = f(x_0) + \boxed{\cancel{Df|_{x_0}(x - x_0)}} + \boxed{\frac{1}{2} D^2 f| \ldots}$

$$f(x) = f(x_0) + \boxed{\cancel{Df|_{x_0} \underbrace{(x-x_0)}_{=:h}}} + \boxed{\frac{1}{2} D^2 f|_{x_0}(h,h)} + \dots$$

$$\dots + \frac{1}{r!} D^r f|_{x_0}(h,\dots,h) + o(\|h\|_V^r)$$

Def.: Let X be a top. space and $f: X \to \mathbb{R}$. A point $x_0 \in X$ is called a (strict) local maximum of f, iff $\exists U \subset \mathcal{U}(x_0)$ s.t.

$$\forall x \in U \setminus \{x_0\} : f(x) \underset{<}{\leq} f(x_0)$$

loc. minimum analogous.

Thm.: Let $G \subset V$ and $f \in C^1(G, \mathbb{R})$ have a local extremum at $x_0 \in G$. Then $Df|_{x_0} = 0$.

Thm.: Let $G \subset V$ and $f \in C^2(G, \mathbb{R})$ and $x_0 \in G$ such that $Df|_{x_0} = 0$.

(a) If $D^2 f|_{x_0}(h,h) > 0 \quad \forall h \in V \setminus \{0\}$, then f has a strict local minimum at x_0

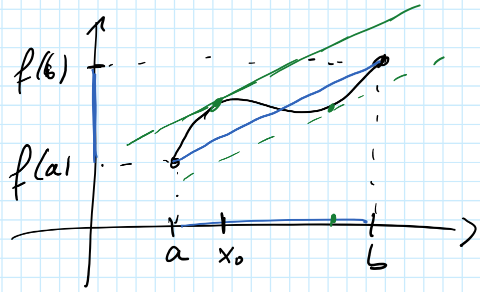
(b) $D^2 f|_{x_0}(h,h) < 0 \ \forall h \in V \setminus \{0\} \Rightarrow$ strict. loc. max. at x_0

(c) If $D^2 f|_{x_0}$ is indefinite, then f has no local extremum at x_0

Mean value thm: $f:[a,b]\to\mathbb{R}$ cont. and

cont. differentiable on (a,b). Then $\exists\, x_0\in(a,b)$:

$$f'(x_0) = \frac{f(b)-f(a)}{b-a}$$



$$\Rightarrow \boxed{f(b)-f(a) = f'(x_0)(b-a)}$$

Thm: Let $G\subset\mathbb{R}^n$ and $f: G\to\mathbb{K}^m$ cont. diff.

Let $\gamma:[a,b]\to G$ cont. diff. Then

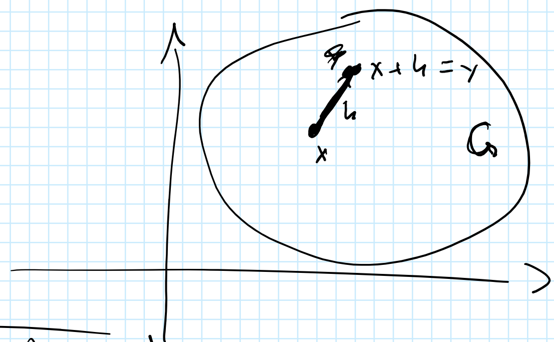
$$\boxed{f(\gamma(b)) - f(\gamma(a)) = \int_a^b \underbrace{Df\big|_{\gamma(t)}}_{\mathbb{K}^m\leftarrow\mathbb{R}^n}\cdot\underbrace{\gamma'(t)}_{\in\mathbb{R}^n}\,dt}$$

Thm (mean value thm)

$G\subset\mathbb{R}^n$, $f\in C^1(G,\mathbb{K}^m)$.

Let $x\in G$ and $h\in\mathbb{R}^n$ s.t.

$\{x+th \mid t\in[0,1]\}\subset G$. Then



$$\boxed{f(x+h)-f(x) = \left(\int_0^1 Df\big|_{x+th}\,dt\right)\cdot h}$$

$$\Rightarrow \|f(x)-f(y)\| \le \underbrace{\left\|\int_0^1 Df\big|_{x+th}\,dt\right\|_{op}}_{\sup_{z\in\overline{xy}}\|Df|_z\|}\cdot\|x-y\|$$

For $f:\mathbb{R}^n\to\mathbb{R}$ we obtain again $\exists\, z\in\overline{xy}$

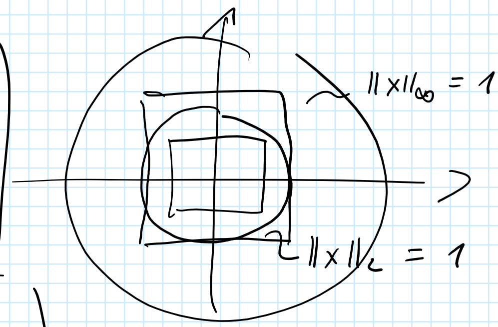
$$f(y)-f(x) = Df\big|_z\cdot(y-x)$$

Def: Equivalence of norms

Two norms $\|\cdot\|_a$ and $\|\cdot\|_b$ on a vectorspace V are **equivalent**, iff $\exists\, c, C > 0 \ \forall x \in V$

$$c\,\|x\|_a \leq \|x\|_b \leq C\,\|x\|_a$$

(equivalence rel.)



Thm: On a finite dim. vector space all norms are equivalent.

Thm: All finite dim. normed spaces are complete.

Def: Fréchet derivative

Let X and Y are Banach spaces and $G \subset X$ open.

A map $f : G \to Y$ is differentiable at $x \in G$, iff

$\exists$ continuous linear map $A : X \to Y$ s.t.

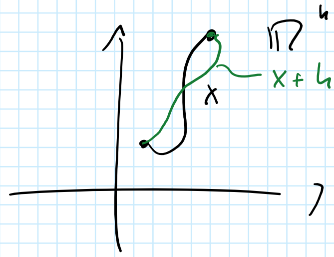
$$f(x+h) = f(x) + Ah + o(\|h\|_X)$$

for h in a neighbourhood of $0 \in X$. $A = Df|_x$

Example: Euler-Lagrange equations

$X = C^2([0,T], \mathbb{R}^n)$ $\quad x \in X,\ x : [0,T] \to \mathbb{R}^n$

$$\boxed{\|x\|_X := \underbrace{\|x\|_\infty}_{= \sup_{t\in[0,T]} \|x(t)\|_2} + \|\dot{x}\|_\infty + \|\ddot{x}\|_\infty}$$



$S : X \to \mathbb{R}$ (action)

$$\boxed{x \mapsto S(x) := \int_0^T L(x(t), \dot{x}(t))\, dt}$$

where $L: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $(q, v) \mapsto L(q, v) \quad \in C^2$

(e.g. $L(q,v) := \frac{1}{2} m \|v\|^2 - V(q)$)

We compute the derivative $DS|_x$ of S: $x, h \in X$

$$S(x+h) = \int_0^T L(x(t)+h(t), \dot{x}(t) + \dot{h}(t))\, dt$$

$$= \int_0^T \Big(L(x(t), \dot{x}(t)) + \big\{ D_q L|_{(x(t), \dot{x}(t))} \cdot h(t) + D_v L|_{(x(t),\dot{x}(t))} \cdot \dot{h}(t) \big\} \Big)\, dt + \cancel{o(\|h\|_X)} \quad \mathcal{O}(\|h\|_X^2)$$

$$= S(x) + \underbrace{\int_0^T \Big(D_q L|_{(x(t),\dot{x}(t))} h(t) + D_v L|_{(x(t), \dot{x}(t))} \dot{h}(t) \Big)\, dt}_{= DS|_x \cdot h} + \mathcal{O}(\|h\|_X^2)$$

$\Rightarrow$

$$= S(x) + \cancel{D_v L|_{(x(T), \dot{x}(T))} \cdot h(T)} - \cancel{D_v L|_{(x(0), \dot{x}(0))} \cdot h(0)}$$

$$+ \int_0^T \Big(\underbrace{D_q L|_{(x(t), \dot{x}(t))} - \Big(\frac{d}{dt} D_v L|_{(x(t), \dot{x}(t))}\Big)}_{=0} \Big) h(t)\, dt$$

$$+ \mathcal{O}(\|h\|_X^2)$$

$= DS|_x\, h$

for $h \in X$ s.t. $h(0) = h(T) = 0$

for $h \in X$ s.t. $h(0) = h(T) = 0$

$$DS|_x h = 0 \iff \boxed{D_q L\Big|_{(x(t),\dot{x}(t))} - \frac{d}{dt} D_v L\Big|_{(x(t),\dot{x}(t))} = 0}$$

Euler-Lagrange eqn.

$$-D_q V(x(t)) - m\ddot{x}(t) = 0$$

$f : V_1 \times V_2 \to W$ $\qquad (v_1, v_2) \in V_1 \times V_2$

$f_{v_2} : V_1 \to W \quad , \quad f_{v_2}(v_1) = f(v_1, v_2)$

$D_{v_1} f|_{(v_1,v_2)} := D f_{v_2}|_{v_1}$

$f : \mathbb{R}_x^n \times \mathbb{R}_y^m \to \mathbb{K}^l$

$$Df = \Bigg(\underbrace{\begin{matrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ & \vdots & \\ \frac{\partial f_l}{\partial x_1} & \cdots & \cdot \end{matrix}}_{D_x f} \;\; \underbrace{\begin{matrix} \frac{\partial f_1}{\partial y_1} & \cdots & \frac{\partial f_1}{\partial y_m} \\ & & \\ - & - & \frac{\partial f_l}{\partial y_m} \end{matrix}}_{D_y f}\Bigg)$$

Def.: A subset $A \subset X$ of a top. space $(X, \mathcal{T})$ is called dense, iff $\overline{A} = X$.

Rem: If $A \subset X$ is dense, then $A \cap O \neq \emptyset$ for any $O \in \mathcal{T} \setminus \{\emptyset\}$

Prop.: Let $f, g : X \to Y$ be cont. fcts., Y Hausdorff, and $A \subset X$ dense. Then

$$f|_A = g|_A \implies f = g.$$

Exc: prove this.