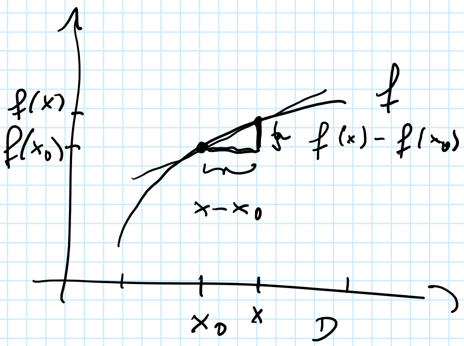
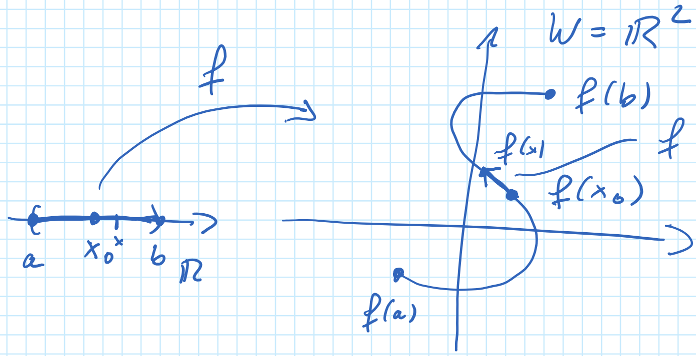
Recall that for $f: \mathbb{R} \supset D \longrightarrow \cancel{\mathbb{R}}\ (W, \|\cdot\|)$

$$f'(x_0) := \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

[$\forall (x_n)$ in $D \setminus \{x_0\}$ with $\lim_{n\to\infty} x_n = x_0$

$\lim_{n\to\infty} \frac{f(x_n) - f(x_0)}{x_n - x_0}$ exists.]





$f'(x_0)$ = slope of tangent to the graph at $(x_0, f(x_0))$

$f'(x_0) \in W$ = velocity vector

Def.: Let $n \in \mathbb{N}$, $D \subset \mathbb{R}^n$ open, $(W, \|\cdot\|)$ a normed space. For $x \in D$ and $j \in \{1, .., n\}$ a function $f: D \to W$ is called <u>partially differentiable</u> in the jth coord. direction at x, iff the limit

$$\lim_{h \to 0} \frac{f(x + h e_j) - f(x)}{h} = \lim_{h \to 0} \frac{f(x_1, x_2, .., x_j + h, .., x_n) - f(x_1, \dots, x_n)}{h}$$

exists. One writes

$$\frac{\partial f}{\partial x_j}(x) = \partial_j f(x) := \lim_{h \to 0} \frac{f(x + h e_j) - f(x)}{h}$$

and calls the vector $\partial_j f(x) \in W$ the jth <u>partial</u>

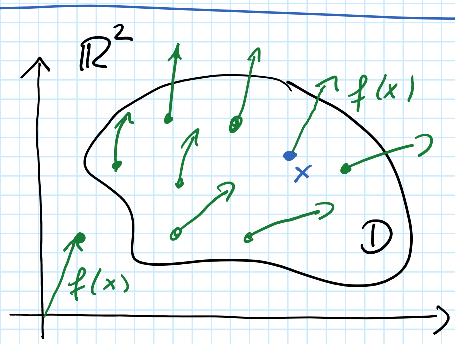
and calls the vector $\partial_j f(x) \in W$ the jth partial derivative at x.

If f is partially diff. and the partial derivatives $\partial_j f : D \to W$ are cont. fcts., then f is called cont. part. diff. The vector space of c.p.d. fcts on $D \subset \mathbb{R}^n$ is denoted by $C^1(D, W)$.

The gradient of f at x is

$$\nabla f(x) := (\partial_1 f(x), \ldots, \partial_n f(x)) \in W^n$$

Def.: A map $f : \mathbb{R}^n \supset D \to \mathbb{R}^n$ is called a vector field.



Ex.: The gradient $\nabla f : D \to \mathbb{R}^n$ of a fct. $f : D \to \mathbb{R}$, $D \subset \mathbb{R}^n$, defines a vector field.

Def.: A fct. $f : \mathbb{R}^n \supset D \to W$ ($\in$ $\mathbb{K}$-VSp.) is called r-times cont. part. diff., iff for all $j = (j_1, \ldots, j_r)$, $j_i \in \{1, \ldots, n\}$

- f is c.p.d.
- $\partial_{j_1} f$ is c.p.d.
- $\partial_{j_2} \partial_{j_1} f$ is c.p.d.

$\vdots$

- $\partial_{j_1} f$ is c.p.d.
- $\partial_{j_2}\partial_{j_1} f$ is c.p.d.

$\vdots$

- $\partial_{j_r}\cdots\partial_{j_1} f$ is continuous

The $\mathbb{K}$-vector space of r-times c.p.d. fcts is denoted by $C^r(D,W)$.

<u>Def.:</u> Let $D\subset\mathbb{R}^n$, $g\in C^1(D,\mathbb{R}^n)$, $f\in C^2(D,\cancel{\mathbb{R}}\,W)$.

Then
$$\operatorname{div} g : D\to\mathbb{R},\quad x\mapsto \operatorname{div} g(x) := \sum_{j=1}^n \frac{\partial g_j}{\partial x_j}(x)$$

is called the <u>divergence</u> of g,

$$\operatorname{curl} g : D\to\mathbb{R}^3,\quad x\mapsto \operatorname{curl} g(x) := \begin{pmatrix} \partial_2 g_3(x)-\partial_3 g_2(x)\\ \partial_3 g_1(x)-\partial_1 g_3(x)\\ \partial_1 g_2(x)-\partial_2 g_1(x)\end{pmatrix}$$

for $n=3$ is called the <u>curl</u> of g,

and

$$\Delta f : D\to \cancel{\mathbb{R}}\,W,\quad x\mapsto \Delta f(x) := \big(\operatorname{div}(\nabla f)(x)\big) = \sum_{j=1}^n \frac{\partial^2 f}{\partial x_j^2}(x)$$

is called <u>Laplace</u> of f.

<u>Ex. 1:</u> Let $g:\mathbb{R}^n\to\mathbb{R}^n$, $x\mapsto g(x)=x$ $\quad(g=id)$

and $f:\mathbb{R}^n\setminus\{0\}\to\mathbb{R}$, $x\mapsto \|x\|_2$.

Compute $\operatorname{div}(g)$ and Δf.

<u>Thm.</u> (Schwarz)

Let $f \in C^2(D, W)$, $D \subset \mathbb{R}^n$.

Then $\forall x \in D$, $j, i \in \{1, \dots, n\}$

$$\partial_j \partial_i f(x) = \partial_i \partial_j f(x)$$

<u>Corollary</u>: Let $D \subset \mathbb{R}^3$, $f \in C^2(D, \mathbb{R})$ and $g \in C^2(D, \mathbb{R}^3)$.

Then $\operatorname{curl}(\nabla f) = 0$ and $\operatorname{div}(\operatorname{curl} g) = 0$

<u>Def.</u>: Let V be a <u>real</u> normed space, W a normed space, $D \subset V$ open and $f : D \to W$.

The <u>directional derivative</u> of f at $x \in D$ in the direction $v \in V$ is

$$\partial_v f(x) := \lim_{h \to 0} \frac{f(x + hv) - f(x)}{h} = \underline{\frac{d}{dh} f(x + hv)\Big|_{h=0}}$$

if the limit exists.

<u>Example</u>: For $f : \mathbb{R}^n \to W$ we have $\partial_{e_j} f(x) = \partial_j f(x)$.

<u>Different viewpoint: Derivative as lin. approx.</u>

For $f : \mathbb{R} \to W$ differentiability at $x_0 \in \mathbb{R}$ means

$$0 = \lim_{x \to x_0} \left(\frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \right) =: \lim_{x \to x_0} \frac{\varphi(x, x_0)}{x - x_0}$$

where $\varphi(x, x_0) = f(x) - f(x_0) - f'(x_0)(x - x_0)$

or, after reshuffling

$$\boxed{f(x) = \underbrace{f(x_0) + f'(x_0) \cdot (x - x_0)}_{(*)} + \varphi(x, x_0)}$$

where

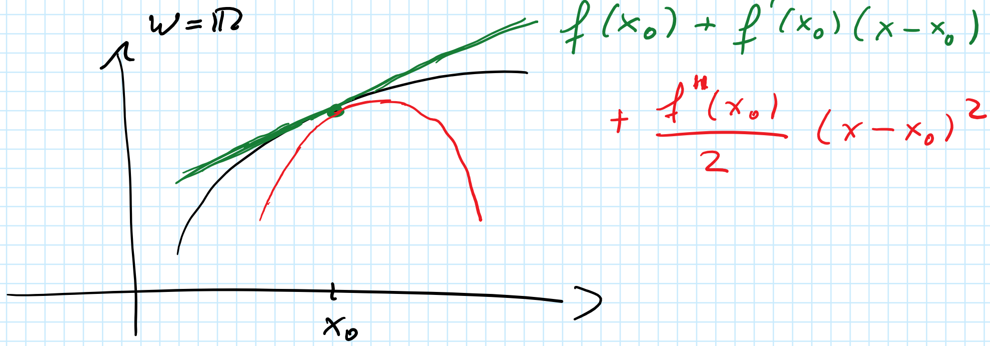
$$\varphi(x, x_0) = o(|x - x_0|) :\Longleftrightarrow \lim_{x \to x_0} \frac{|\varphi(x, x_0)|}{|x - x_0|} = 0$$

The map $\mathbb{R} \to W$, $x \mapsto f'(x_0) \cdot x$ is $\mathbb{R}$-linear

and the map $\mathbb{R} \to W$, $x \mapsto f(x_0) + f'(x_0)(x - x_0)$

is affine-$\mathbb{R}$-linear.

Hence, we think of $(*)$ as the (affine) linear approximation to f near x_0.



Def.: Total derivative

Let V be a finite dim. real vector space, W a normed space, $G \subset V$ open and $f: G \to W$. f is <u>differentiable</u> at $x_0 \in G$, iff there exist an $\mathbb{R}$-linear map $A: V \to W$ such that

$$\lim_{x \to x_0} \frac{f(x) - f(x_0) - A(x - x_0)}{\|x - x_0\|_V} = 0$$

(here $\|\cdot\|_V$ is any norm on V)

Then A is uniquely determined (Ex. 2: prove this!), is denote by $Df|_{x_0}$, and called the total derivative or the differential of f at x_0.

If $f: G \to W$ is diff. at all $x \in G$, then f is called <u>differentiable on G</u> and

$$Df : G \to \mathcal{L}(V, W), \quad x \mapsto Df|_x$$

is a fct. on G taking values in the linear maps from V to W.

Rem.: $f: G \to W$ is diff. at x_0

$$\Rightarrow \quad \boxed{f(x) = f(x_0) + Df|_{x_0} \underbrace{(x - x_0)}_{\to 0} + \underbrace{o(\|x - x_0\|_V)}_{\substack{\to 0 \\ \text{faster!}}}}$$

Example: Let $L: V \to W$ be an $\mathbb{R}$-lin. map.

Then $L(x) = L(x_0 + x - x_0) = L(x_0) + L(x - x_0)$

$=: L(x_0) + DL|_{x_0}(x - x_0)$

and hence $DL|_{x_0} = L$.

$f(x) = x \quad , \quad f'(x) = 1 \quad ?$

Thm.: Let $G \subset \mathbb{R}^n$ and let $f: G \to \mathbb{K}^m$ be differentiable at $x_0 \in G$. Then

$$\left(Df|_{x_0}\right)_{ij} = \frac{\partial f_i}{\partial x_j}(x_0)$$

or, more explicitly,

$$Df|_{x_0} = \underbrace{\begin{pmatrix} \partial_1 f_1(x_0) & \cdots & \partial_n f_1(x_0) \\ \vdots & & \\ \partial_1 f_m(x_0) & \cdots & \partial_n f_m(x_0) \end{pmatrix}}_{n \text{ columns}} \Bigg\} m \text{ rows}$$

$$= \begin{pmatrix} \nabla f_1(x_0) \\ \vdots \\ \nabla f_m(x_0) \end{pmatrix}$$ „Jacobi matrix"

Thm.: Let $G \subset \mathbb{R}^n$ open and $f \in C^1(G, \mathbb{K}^m)$. Then f is differentiable.

cont. part. diff.

cont. part. diff. $\Rightarrow$ differentiable $\Rightarrow$ part. diff.

$\Downarrow$

continuous

None of the implications holds in the reversed direct.!
But

cont. part. diff $\iff$ diff. with cont. derivative

Thm.: Chain rule

Let U, V be finite dim. real vector spaces, W a normed space, $G \subset U$, $H \subset V$ open and $g: G \to V$, $f: H \to W$ maps with $g(G) \subset H$, i.e.

$$U \supset G \xrightarrow{g} H \subset V \xrightarrow{f} W.$$

If g is diff. at $x \in G$ and f is diff. at $g(x) \in H$, then $f \circ g: G \to W$ is diff. at x and

$$D(f \circ g)|_x = \underbrace{Df|_{g(x)}}_{W \leftarrow V} \cdot \underbrace{Dg|_x}_{V \leftarrow U}$$

Corollary: For $f \in C^1(G, W)$, $G \subset V$, $x_0 \in G$, and $v \in V$ we have

$$\partial_v f(x_0) = Df|_{x_0} v$$

Ex. 3: prove this!

For $f: G \to W$ the differential Df is a map ($G \subset V$)

$Df: G \to \mathcal{L}(V, W)$. Thus the „second derivative"

$D(Df)$ is a map

$$D(Df) : G \to \mathcal{L}(V, \mathcal{L}(V,W)) \cong \mathcal{L}_2(V \times V, W)$$

$:=$ bilinear maps $V \times V \to W$

and the rth derivation

$$D^r f : G \to \mathcal{L}_r(\underbrace{V \times \cdots \times V}_{r\text{-copies}}, W).$$

<u>Thm.</u> <u>Taylor</u>

Let $G \subset V$ open, $x_0 \in G$, and $\delta > 0$ s.t. $B_\delta(x_0) \subset G$.

Then for any $f \in C^r(G, W)$ and $x \in B_\delta(x_0)$

$$f(x) = f(x_0) + Df|_{x_0} \underbrace{(x - x_0)}_{=: h} + \frac{1}{2} D^2 f|_{x_0}(h,h) + \dots$$

$$\dots + \frac{1}{r!} D^r f|_{x_0}(h, \dots, h) \quad + o(\|h\|_V^r)$$

<u>Def.</u>: Let X be a top. space and $f: X \to \mathbb{R}$. A point $x_0 \in X$ is called a (strict) local maximum of f, iff $\exists U \subset \mathcal{U}(x_0)$ s.t.

$$\forall x \in U \setminus \{x_0\} : f(x) \underset{<}{\leq} f(x_0)$$

loc. minimum analogous.

<u>Thm.</u>: Let $G \subset V$ and $f \in C^1(G, \mathbb{R})$ have a local extremum at $x_0 \in G$. Then $Df|_{x_0} = 0$.

Thm.: Let $G \subset V$ and $f \in C^1(G, \mathbb{R})$ have a local extremum at $x_0 \in G$. Then $Df|_{x_0} = 0$.

Thm.: Let $G \subset V$ and $f \in C^2(G, \mathbb{R})$ and $x_0 \in G$ such that $Df|_{x_0} = 0$.

(a) If $D^2f|_{x_0}(h,h) > 0 \quad \forall h \in V \setminus \{0\}$, then f has a strict local minimum at x_0

(b) $D^2f|_{x_0}(h,h) < 0 \ \forall h \in V \setminus \{0\} \Rightarrow$ strict. loc. max. at x_0

(c) If $D^2f|_{x_0}$ is indefinite, then f has no local extremum at x_0