Implicit fcts:

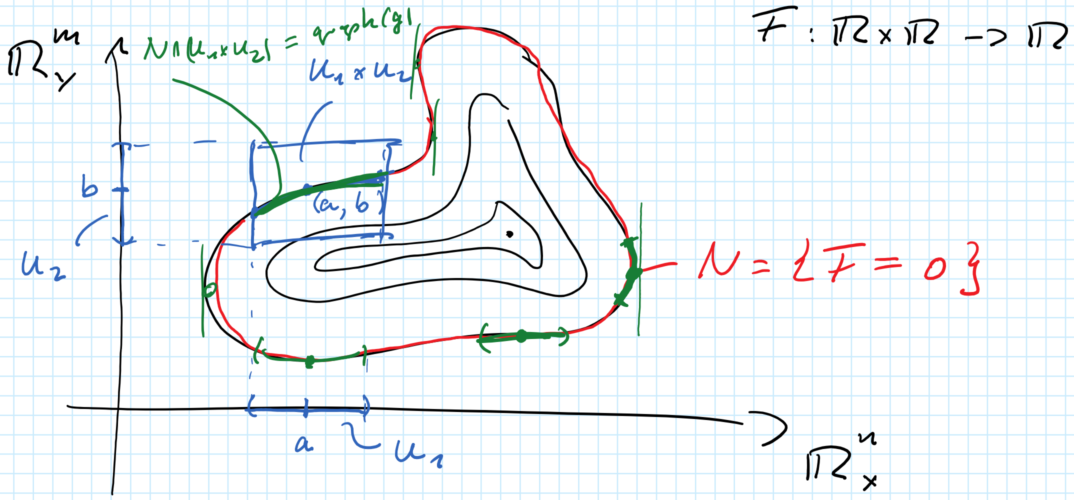
When is it possible to smoothly parametrize the level sets of a function

$$\boxed{F : \underbrace{\mathbb{R}^n \times \mathbb{R}^m}_{\mathbb{R}^{n+m}} \to \mathbb{R}^m}, \quad (x,y) \mapsto F(x,y) \quad ?$$

If F is „smooth“, then we expect the level sets to be n-dim. submanifolds of $\mathbb{R}^{n+m}$, i.e. sets that locally look like the graph of a smooth fct. $g : \mathbb{R}^n \to \mathbb{R}^m$.

$n=1 \quad m=1$

$F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$



$$\boxed{\begin{array}{l} \underline{\text{Implicit function theorem}} \\ \text{Let } G \subset \mathbb{R}^{n+m} \text{ be open, } F \in C^1(G, \mathbb{R}^m) \text{, and} \\ N := \{(x,y) \in G \mid F(x,y) = 0\}. \\ \text{If for } (a,b) \in N \text{ it holds that the matrix} \\ D_y F\big|_{(a,b)} := \begin{pmatrix} \frac{\partial F_1}{\partial y_1} & \cdots & \frac{\partial F_1}{\partial y_m} \\ \end{pmatrix} \end{array}}$$

$$D_y F|_{(a,b)} := \begin{pmatrix} \frac{\partial F_1}{\partial y_1} & \cdots & \frac{\partial F_1}{\partial y_m} \\ \vdots & & \\ \frac{\partial F_m}{\partial y_1} & \cdots & \frac{\partial F_m}{\partial y_m} \end{pmatrix}_{(a,b)}$$

is invertible, then there exist open neighbourhoods $U_1 \subset \mathbb{R}^n$ of a and $U_2 \subset \mathbb{R}^m$ of b with $U_1 \times U_2 \subset G$ and a fct. $g \in C^1(U_1, U_2)$ such that

$$N \cap (U_1 \times U_2) = \text{graph}(g)$$

(more explicitly: $\forall (x,y) \in U_1 \times U_2$: $F(x,y) = 0 \Leftrightarrow g(x) = y$)

($\Leftrightarrow$) one can solve $F(x,y) = 0$ locally for $y = g(x)$

$F(x, g(x)) = 0$)

Moreover,

$$\boxed{D g|_x = -\left(D_y F|_{(x,g(x))}\right)^{-1} \cdot D_x F|_{(x,g(x))}}$$

(Apply chain rule to $D F(x, g(x)) = 0$)

Def.: Let $G, H \subset \mathbb{R}^n$ be open. A map $C^1(G, H)$ is called a diffeomorphism, iff it is bijective and also $f^{-1} \in C^1(H, G)$.

Exc 1: Prove that the differential $Df|_x : \mathbb{R}^n \to \mathbb{R}^n$ of a diffeom. f is invertible for all $x \in G$.

Sol. 1: $f : \mathbb{R}^n \supset G \to H \subset \mathbb{R}^n$ diffeom.

$f^{-1}: H \to G \in C^1(H,G)$

$f^{-1} \circ f = id_G \rightsquigarrow id_G = D(f^{-1} \circ f)|_x = Df^{-1}|_{f(x)} \cdot Df|_x$

$\Rightarrow D(f^{-1})|_{f(x)} = (Df|_x)^{-1}$

Ex 2: Give an example of a bijection $f \in C^1$ such that f^{-1} is not cont. differentiable.

Sol. 2: $f: \mathbb{R} \to \mathbb{R}$, $x \mapsto x^3$ $\quad f \in C^\infty(\mathbb{R})$

$f^{-1}: \mathbb{R} \to \mathbb{R}$, $x \mapsto \sqrt[3]{x}$ satisfies $f \circ f^{-1} = f^{-1} \circ f = id_{\mathbb{R}}$.

but f^{-1} is not differentiable at 0. ($f'(0) = 0$) does not apply

Inverse fct. theorem

Let $G \subset \mathbb{R}^n$ be open and $f \in C^1(G, \mathbb{R}^n)$. If for $a \in G$ it holds that $Df|_a$ is invertible then there exists an open neighbourhood U of a such that $f|_U : U \to f(U) \subset \mathbb{R}^n$ is a diffeomorphism.

Example: $\exp: \mathbb{R} \to (0,\infty)$

$(\exp)^{-1} = \ln : (0,\infty) \to \mathbb{R}$

Since ~~$(\exp)' = \exp$~~ $(e^x)' = e^x \neq 0 \ \forall x \in \mathbb{R}$

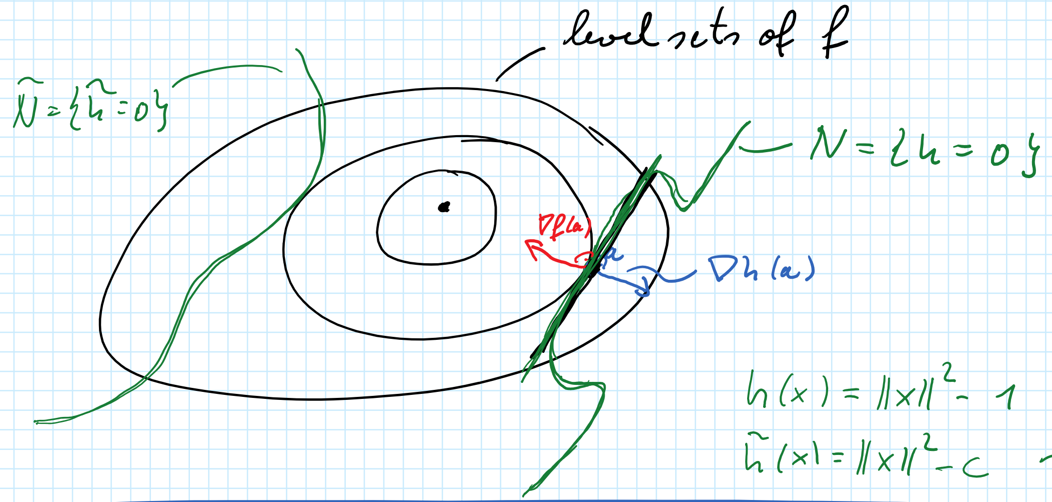
exp is a diffeomorphism

Def.: Let $G \subset \mathbb{R}^n$ be open and $f, h \in C^1(G, \mathbb{R})$.
Let $N := \{x \in G \mid h(x) = 0\}$ and $a \in N$.
We say that f has a local extremum (max. or min) at a under the constraint $h = 0$, iff $f|_N$ has a local extremum at a.

Thm.: Let G, f, h, N as above. If $a \in N$ is a regular point of h (i.e. $\nabla h|_a \neq 0$) and a local extremum of f under the constraint $h = 0$, then there exists $\lambda \in \mathbb{R}$ s.t.

$$\nabla f(a) = \lambda \nabla h(a) \qquad (*)$$

λ = Lagrange parameter



$h(x) = \|x\|^2 - 1 \rightsquigarrow S^{d-1} = \{h = 0\}$
$\tilde{h}(x) = \|x\|^2 - c \rightsquigarrow S^{d-1} = \{\tilde{h} = 1\}$

Thm.: Let $G \subset \mathbb{R}^n$ be open, $f, h \in C^2(G, \mathbb{R})$.
Let for $a \in N$ the necessary cond. $(*)$ be satisfied, i.e. there exists $\lambda \in \mathbb{R}$ s.t. $\nabla F_\lambda(a) := \nabla(f - \lambda h)(a) = 0$.

i.e. there exists $\lambda \in \mathbb{R}$ s.t. $\nabla f_\lambda(a) := \nabla(f - \lambda h)(a) = 0$.

(a) If $D^2 F|_a (v,v) > 0$ for all $v \in \mathbb{R}^n \setminus \{0\}$

$Dh|_a v = 0$, then f has a strict local minimum at a under the constraint $h = 0$.

(b) + (c) analogously.

Rem: If $h : G \subset \mathbb{R}^n \to \mathbb{R}^k$, then $N = \{h = 0\}$ is a $(n-k)$-dimensional submanifold

In this case the necessary cond. for extremum under constraint N becomes

$$\boxed{Df(a) \in \operatorname{span}\{Dh_1(a), Dh_2(a), \dots, Dh_k(a)\}}$$

$$\iff \exists \lambda \in \mathbb{R}^k : \boxed{D(f - \lambda \cdot h)|_a = 0}$$

$$Df = \lambda \cdot Dh$$

$$= \lambda_1 Dh_1 + \dots + \lambda_k Dh_k$$

Ordinary differential equations

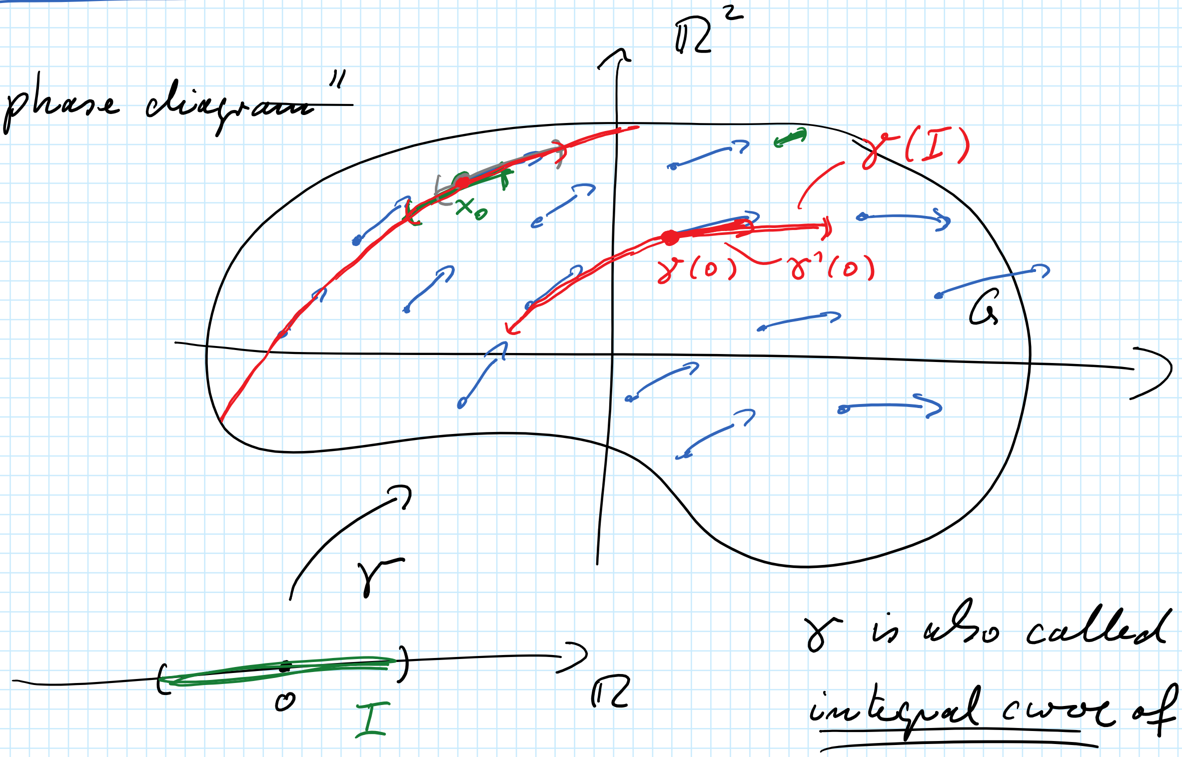
Def.: Let $G \subset \mathbb{R}^n$ open, $v \in C(G, \mathbb{R}^n)$ a continuous vector field, and $I \subset \mathbb{R}$ an open interval containing $0 \in \mathbb{R}$. A curve $\gamma \in C^1(I, G)$ is a solution of the autonomous first order ODE

$$\boxed{\gamma' = v(\gamma)}$$ with initial datum $x_0 \in G$

$\gamma = D(\gamma)$ with initial datum $x_0 \in G$

iff $\gamma' = v \circ \gamma$, i.e. $\gamma'(t) = v(\gamma(t))\ \forall t \in I$, and $\gamma(0) = x_0$.

„phase diagram"

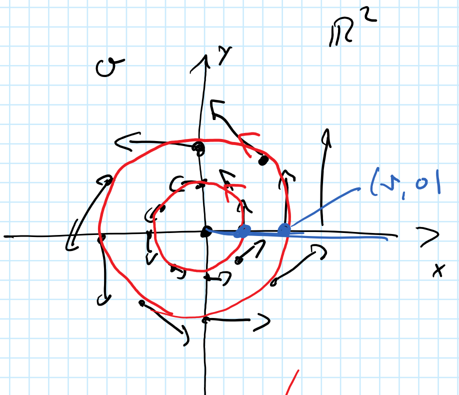


γ is also called an <u>integral curve</u> of v

A $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ $=$

<u>Ex. 3.:</u> Determine and draw some integral curves for the vector fields $\underline{v}: \mathbb{R}^2 \to \mathbb{R}^2,\ (x,y) \mapsto v(x,y) = \begin{pmatrix} -y \\ x \end{pmatrix}$

$\underline{\omega}: \mathbb{R}^2 \to \mathbb{R}^2,\ (x,y) \mapsto \omega(x,y) = \begin{pmatrix} x \\ y \end{pmatrix}$

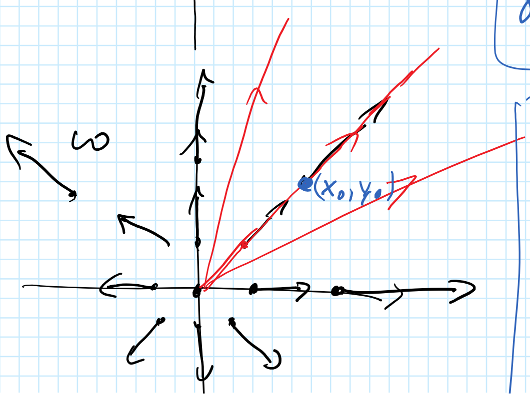


$$\boxed{\gamma_v(t) = r \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}}$$

$\gamma_v(0) = r\,(1,0) = (r,0)$

$r \geq 0$

$$\boxed{\gamma_v'(t) = r \begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix} = \underline{v}(\gamma(t))}$$



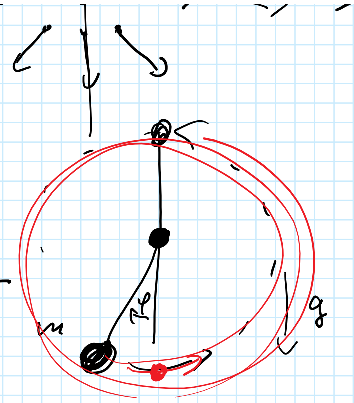
$$\gamma_\omega(t) = e^t \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$\gamma_\omega(0) = (x_0, y_0)$

$$\boxed{\gamma_\omega'(t) = \gamma_\omega(t) = \underline{\omega}(\gamma_\omega(t))}$$

$$\boxed{\gamma_\omega'(t) = \gamma_\omega(t) = \omega(\gamma_\omega(t))}$$

Ex.:



$$\boxed{\ddot{\varphi}(t) = -c \sin(\varphi(t))}$$

$$\begin{pmatrix} \dot{\varphi} \\ \dot{\omega} \end{pmatrix} = \begin{pmatrix} \omega \\ -c \underbrace{\sin(\varphi)} \end{pmatrix} = v((\varphi, \omega))$$

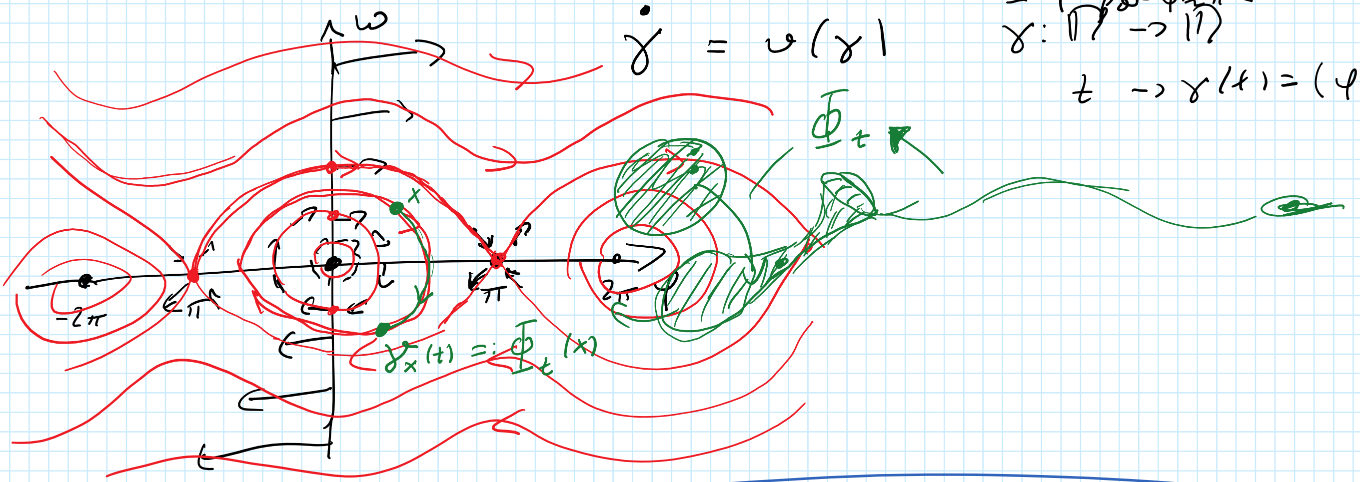
$\approx \varphi$ for $\varphi \approx 0$

$\approx -\varphi$ for $\varphi \approx \pi$?

$\gamma : \mathbb{R} \to \mathbb{R}^2$

$t \to \gamma(t) = (\varphi(t), \omega(t))$

$$\dot{\gamma} = v(\gamma)$$



Def.: Let $m \in \mathbb{N}$. An autonomous mth order ODE on a domain $D \subset \mathbb{R}^n$ is given by a cont. fct.

$$f : D \times \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{(m-1)\text{ copies}} \to \mathbb{R}^n$$

and the equation

$$\gamma^{(m)} = f(\gamma, \gamma', \gamma'', \dots, \gamma^{(m-1)}).$$

Now $\gamma \in C^m(I, D)$ is called a solution with initial datum $(x_0, y_1, \dots, y_{m-1})$, iff

$$\gamma^{(m)}(t) = f(\gamma(t), \gamma'(t), \dots, \gamma^{(m-1)}(t)) \quad \forall t \in I$$

$$\gamma^{(n)}(t) = f(\gamma(t), \gamma'(t), \dots, \gamma^{(n-1)}(t)) \quad \forall t \in I$$

and $\gamma(0) = x_0$ and $\gamma^{(j)}(0) = y_j \quad \forall j = 1, \dots, n-1$.

<u>Def.:</u> Let $J \subset \mathbb{R}$ be an open interval. A cont. map

$$v : J \times D \to \mathbb{R}^n, \quad (t,x) \mapsto v(t,x)$$

is called a <u>time-dependent vector field</u>. The ODE

$$\boxed{\gamma' = v(t, \gamma)}$$

is called a <u>non-autonomous first order ODE</u>.

If $I \subset J$ is an open subinterval, $t_0 \in I$, $x_0 \in D$, then $\gamma : I \to D$ is a <u>solution</u> with initial value x_0 for initial time t_0, iff

$$\gamma'(t) = v(t, \gamma(t)) \quad \forall t \in I$$

and $\gamma(t_0) = x_0$.

All the above types of ODEs can be reduced to autonomous first order ODEs. !

<u>Def.:</u> Let $U \subset \mathbb{R} \times \mathbb{R}^n$ and $v \in C(U, \mathbb{R}^n)$.

$J \times D$

(a) We say that v satisfies a <u>Lipschitz condition</u>, iff there exist $L \geq 0$ such that

$$\forall (t,x), (t,y) \in U : \ \underline{\|v(t,x) - v(t,y)\| \leq L \|x - y\|}.$$

$\forall (t,x),(t,y) \in U: \ \|v(t,x) - v(t,y)\| \leq L\|x-y\|$.

(b) We say that v satisfies a local Lip. cond., iff every $(t,x) \in U$ admits a neighbourhood $V \subset U$ such that $v|_V$ satisfies a Lip. cond.

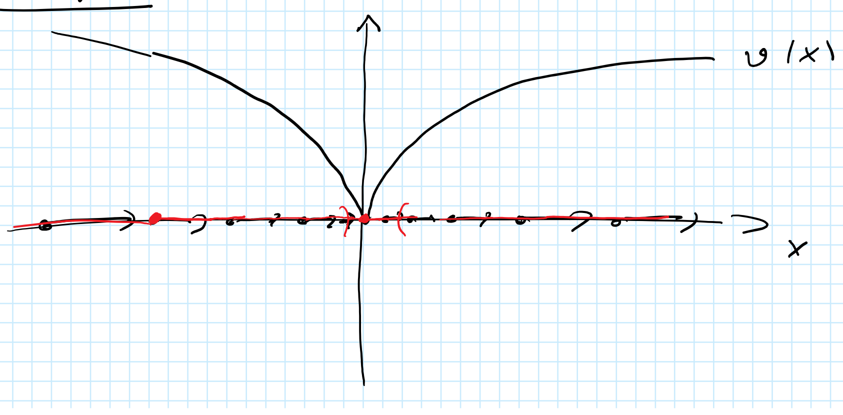
Thm. Picard-Lindelöf

Let $U \subset \mathbb{R} \times \mathbb{R}^n$ be a domain and let $v \in C(U, \mathbb{R}^n)$ satisfy a local Lip. cond.

Local existence: For any $(t_0, x_0) \in U$ there exists $\delta > 0$ and a curve $\gamma \in C^1((t_0 - \delta, t_0 + \delta), \mathbb{R}^n)$ that is a solution of $\gamma' = v(t,\gamma)$ with initial datum $\gamma(t_0) = x_0$.

Uniqueness: If $J \subset \mathbb{R}$ is an interval with $t_0 \in J$ and $\tilde{\gamma}: J \to \mathbb{R}^n$ solves $\gamma' = v(t,\gamma)$ with $\tilde{\gamma}(t_0) = x_0$, then $\tilde{\gamma}(t) = \gamma(t) \ \forall t \in J \cap (t_0 - \delta, t_0 + \delta)$.

Example: $v: \mathbb{R} \to \mathbb{R}$, $x \mapsto v(x) = \sqrt{|x|}$

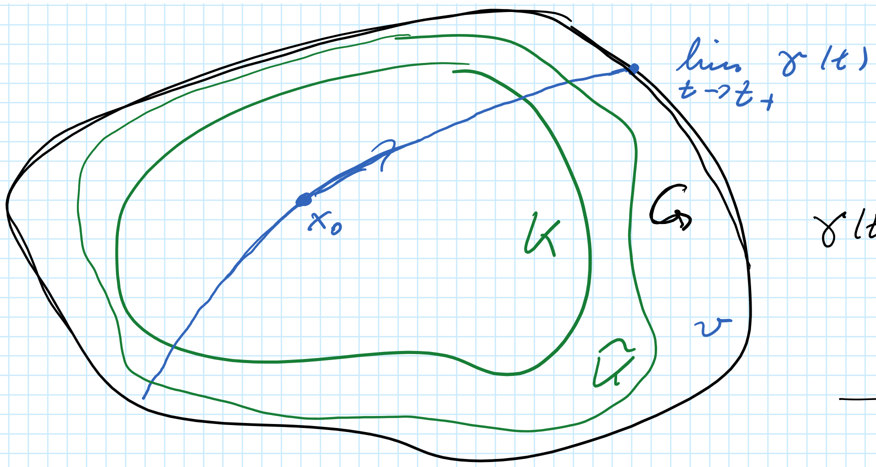


Def.: Let $v \in C(\mathcal{J} \times G, \mathbb{R}^n)$ satisfy a loc. L.c.
A solution $\gamma : I \to G$ of $\gamma' = v(t, \gamma)$ is called
a maximal solution, iff the following holds:
If $I \subset \tilde{I} \subset \mathcal{J}$ and $\tilde{\gamma} : \tilde{I} \to G$ is a sol. of $\gamma' = v(t, \gamma)$
with $\tilde{\gamma}|_I = \gamma$, then $\tilde{I} = I$.

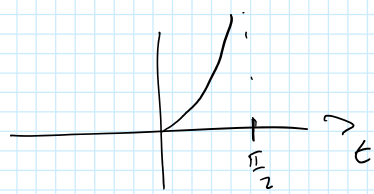
Corollary: Under the cond. of P.L. there exists for
any initial value a unique maximal solution.

Thm.: Let $\underset{\overset{\shortparallel}{\mathbb{R}}}{\mathcal{J}} = (j_-, j_+) \subset \mathbb{R}$, $G \subset \mathbb{R}^n$ a domain,
and $v \in C(\mathcal{J} \times G, \mathbb{R}^n)$ satisfy a loc. Lip. cond.
Let $\gamma : (t_-(t_0, x_0), t_+(t_0, x_0)) \to G$ be the maximal
solution of $\gamma' = v(t, \gamma)$ for the initial value
$(t_0, x_0) \in \mathcal{J} \times G$.
If $t_+(t_0, x_0) < j_+$, then for any compact $K \subset G$
there exists $0 < \tau_K < t_+(t_0, x_0)$ such that

$$\gamma(t) \notin K \quad \text{for all } t \in (\tau_K, t_+(t_0, x_0)).$$



$\gamma(t) = \tan(t)$



<u>Def.:</u> A ^{loc. Lip.} vector field $v \in C(G, \mathbb{R}^n)$ is <u>complete</u>, iff there exists a global solution $\gamma_{x_0} \in C^1(\mathbb{R}, G)$ of $\gamma' = v(\gamma)$ with $\gamma_{x_0}(0) = x_0$ for any initial value $x_0 \in G$.

The associated <u>flow</u> is

$$\Phi : \mathbb{R} \times G \to G, \quad (t, x) \mapsto \Phi(t, x) := \gamma_x(t)$$

and

$$\Phi_t : G \to G, \quad x \mapsto \Phi_t(x) := \Phi(t, x)$$

is called the <u>flow map</u> at time t. It satisfies

$$\Phi_t \circ \Phi_s = \Phi_{t+s} \qquad \forall t, s \in \mathbb{R}$$

i.e. $\mathbb{R} \to \mathrm{Bij}(G \to G)$, $t \mapsto \Phi_t$ is a group action of $(\mathbb{R}, +)$ on the set G.

<u>Thm.:</u> If v satisfies a loc. Lip. cond., then the

$$\boxed{\text{Thm.: If } v \text{ satisfies a loc. Lip. cond., then the corresp. flow maps } \Phi_t : G \to G \text{ are continuous. If } v \in C^1 \text{, then the flow maps } \Phi_t : G \to G \text{ are also } C^1.}$$

continuous resp. differentiable dependence on ~~init~~ initial data.

Linear ODE's

Def: Let $J \subset \mathbb{R}$ open interval, $A : J \to \mathcal{L}(\mathbb{R}^n)$ is cont.

(a) The ODE $\boxed{x' = A(t)\,x}$ $\qquad v(x) = A(t)\,x$

is called a non-autonomous, homogeneous, linear system.

(b) If $b : J \to \mathbb{R}^n$ is continuous, then

$$\boxed{x' = A(t)\,x + b(t)}$$

is called a non-aut., inhomogeneous, linear ODE.

Prop: In the homog. autonomous case

$$\boxed{x' = A\,x}$$

the unique global solution with initial datum $x_0 \in \mathbb{R}^n$ is

$$\boxed{x(t) = e^{At}\,x_0}$$

is
$$\boxed{\gamma(t) = e^{At} x_0}$$

where $e^{At} := \sum_{n=0}^{\infty} \frac{t^n A^n}{n!}$

Exerc. e^{At} for $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$\| $

$\cos(t) \cdot \mathrm{Id} + \sin(t) A$

$\|$

$\begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}$

$A^2 = -\mathrm{Id}$

$A^3 = -A$

$A^4 = \mathrm{Id}$

$A^5 = A$

Thm: $J \subset \mathbb{R}$ open, $A: J \to L(\mathbb{R}^n)$ and $b: J \to \mathbb{R}^n$ cont. Then for every $t_0 \in J$ and $x_0 \in \mathbb{R}^n$ there exists a unique maximal solution $\gamma: J \to \mathbb{R}^n$ of

$$\gamma' = A(t)\gamma + b(t) \qquad \text{with } \gamma(t_0) = x_0$$

Lemma (Grönwall)

Let $a < b$ and $u: [a,b] \to [0,\infty)$ continuous. Assume $\exists\, L, C \geq 0$ such that for $t \in [a,b]$

$$u(t) \leq \cancel{C} + L \int_a^t u(s)\, ds$$

Then

$$u(t) \leq C e^{L(t-a)}$$

Then

$$u(t) \leq C e^{L(t-a)} = 0$$

$\cancel{(t_0 = 0)}$

Def.: Fix $t_0 \in \mathcal{Y}$ and define

int. curve with $\gamma_{x_0}(t_0) = x_0$

$$\Phi_t : \mathbb{R}^n \to \mathbb{R}^n, \quad x_0 \mapsto \gamma_{x_0}(t) \qquad \text{for } t \in \mathcal{Y}$$

the flow map, also called the propagator.

Thm: $\Phi_t : \mathbb{R}^n \to \mathbb{R}^n$ is a linear isomorphism. $(b = 0)$

$\Rightarrow$ The set of solutions $\{ \gamma \in C^1(\mathcal{Y}, \mathbb{R}^n) \mid \gamma' = A(t)\gamma \}$ is an n-dim. subspace of $C^1(\mathcal{Y}, \mathbb{R}^n)$

Thm Let $\Phi_t : \mathbb{R}^n \to \mathbb{R}^n$ be the propagator of a hom. lin. system $\gamma' = A(t)\gamma$ and $b : \mathcal{Y} \to \mathbb{R}^n$ cont. ~~Then~~ Then the solution of the inhomog. eqn.

$$\gamma' = A(t)\gamma + \underbrace{\cancel{b(t)}}_{v(\gamma)} \quad \text{with } \gamma(t_0) = x_0 \text{ is}$$

$$\gamma(t) = \Phi_t \left(x_0 + \int_{t_0}^{t} \Phi_s^{-1} \overset{v(\gamma(s))}{\cancel{b(s)}} \, ds \right)$$

"variation of constants"

$$\gamma' = A\gamma \quad \Rightarrow \quad \Phi_t = e^{A(t-t_0)} = \mathbb{1} + A(t-t_0) + (A(t-t_0))^2/2 + \dots \qquad \swarrow t_0 = 0$$

$$\gamma' = A\gamma \quad \Rightarrow \quad \underline{\underline{\Phi_t = e^{A(t-t_0)}}} = \mathrm{Id} + A(t-t_0) + (A\,t)^2/2 + \dots$$

$$\boxed{\gamma' = A(t)\gamma} \quad \Rightarrow \quad \underline{\Phi}_t = \,:e^{At}: \qquad \text{"} \sum_{j=0}^{\infty} \frac{t^j A^j}{j!} \text{"}$$

$$\underline{\underline{\Phi_t}} = \mathrm{Id} + \int_{t_0}^{t} d\tau\, A(\tau) + \int_{t_0}^{t} d\tau_1 \int_{t_0}^{\tau_1} d\tau_2\, A(\tau_1) A(\tau_2) + \dots$$

$$\tau_2 \leq \tau_1$$

$$= \mathrm{Id} + \sum_{j=1}^{\infty} \underbrace{\int_{t_0}^{t} d\tau_1 \int_{t_0}^{\tau_1} d\tau_2 \cdots \int_{t_0}^{\tau_{j-1}} d\tau_j}\, A(\tau_1) A(\tau_2) \cdots A(\tau_j)$$

All sol. formulas work for bounded linear maps A also on ∞-dim Banachspaces.

only if $[A(t), A(s)] = 0 \;\; \forall t,s \in I$

$$\cancel{t} \curvearrowright \frac{t^2}{2} \curvearrowright \frac{t^3}{2\cdot 3} \curvearrowright$$

$$\boxed{\Phi_t = e^{\int_{t_0}^{t} A(s)\,ds}}$$