Implicit fcts:

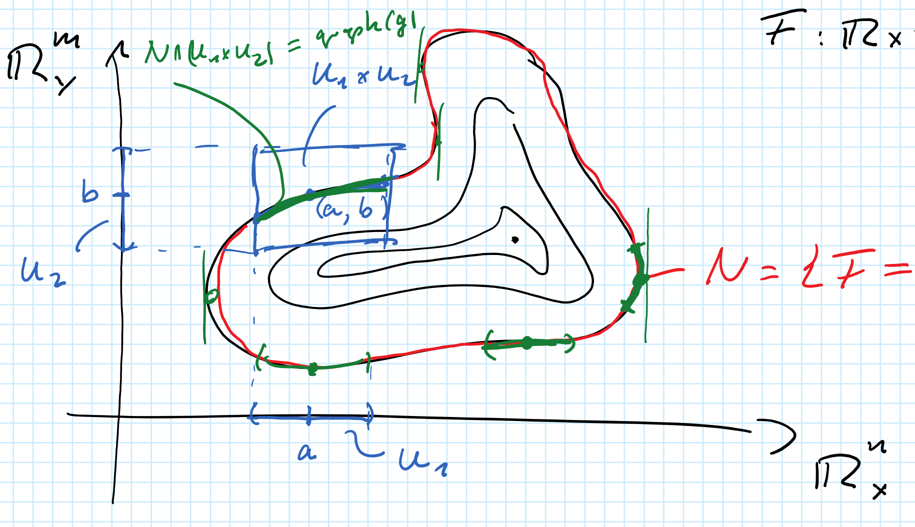
When is it possible to smoothly parametrize the level sets of a function

$$\boxed{F: \underbrace{\mathbb{R}^n \times \mathbb{R}^m}_{\mathbb{R}^{n+m}} \to \mathbb{R}^m}, \quad (x,y) \mapsto F(x,y) \quad ?$$

If F is „smooth", then we expect the level sets to be n-dim. submanifolds of $\mathbb{R}^{n+m}$, i.e. sets that locally look like the graph of a smooth fct. $g: \mathbb{R}^n \to \mathbb{R}^m$.

$n=1 \quad m=1$

$F: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$



Implicit function theorem

Let $G \subset \mathbb{R}^{n+m}$ be open, $F \in C^1(G, \mathbb{R}^m)$, and

$N := \{(x,y) \in G \mid F(x,y) = 0\}$.

If for $(a,b) \in N$ it holds that the matrix

$$\begin{pmatrix} \frac{\partial F_1}{} & \cdots & \frac{\partial F_1}{} \end{pmatrix}$$

of for $(a,b) \in N$ it holds that the matrix

$$D_y F|_{(a,b)} := \begin{pmatrix} \frac{\partial F_1}{\partial y_1} & \cdots & \frac{\partial F_1}{\partial y_m} \\ \vdots & & \\ \frac{\partial F_m}{\partial y_1} & \cdots & \frac{\partial F_m}{\partial y_m} \end{pmatrix}_{(a,b)}$$

is invertible, then there exist open neighbourhoods $U_1 \subset \mathbb{R}^n$ of a and $U_2 \subset \mathbb{R}^m$ of b with $U_1 \times U_2 \subset G$ and a fct. $g \in C^1(U_1, U_2)$ such that

$$N \cap (U_1 \times U_2) = \text{graph}(g)$$

(more explicitly: $\forall (x,y) \in U_1 \times U_2$: $F(x,y) = 0 \Leftrightarrow g(x) = y$)

($\Leftarrow$) one can solve $F(x,y) = 0$ locally for $y = g(x)$

$F(x, g(x)) = 0$)

Moreover,

$$\boxed{D g|_x = -\left(D_y F|_{(x,g(x))}\right)^{-1} \cdot D_x F|_{(x,g(x))}}$$

(Apply chain rule to $D\, F(x, g(x)) = 0$)

<u>Def.</u>: Let $G, H \subset \mathbb{R}^n$ be open. A map $C^1(G, H)$ is called a <u>diffeomorphism</u>, iff it is bijective and also $f^{-1} \in C^1(H, G)$.

<u>Exc 1</u>: Prove that the differential $Df|_x : \mathbb{R}^n \to \mathbb{R}^n$ of a diffeom. f is invertible for all $x \in G$.

Ex 2: Give an example of a bijection $f \in C^1$ such that f^{-1} is <u>not</u> cont. differentiable.

Inverse fct. theorem

Let $G \subset \mathbb{R}^n$ be open and $f \in C^1(G, \mathbb{R}^n)$. If for $a \in G$ it holds that $Df|_a$ is invertible then there exists an open neighbourhood U of a such that $f|_U : U \to f(U) \subset \mathbb{R}^n$ is a diffeomorphism.

Def.: Let $G \subset \mathbb{R}^n$ be open and $f, h \in C^1(G, \mathbb{R})$. Let $N := \{x \in G \mid h(x) = 0\}$ and $a \in N$.
We say that f has a local extremum (max. or min) at a under the constraint $h = 0$, iff $f|_N$ has a local extremum at a.

Thm.: Let G, f, h, N as above. If $a \in N$ is a regular point of h (i.e. $\nabla h|_a \neq 0$) and a local extremum of f under the constraint $h = 0$, then there exists $\lambda \in \mathbb{R}$ s.t.

$$\nabla f(a) = \lambda \nabla h(a) \qquad (*)$$

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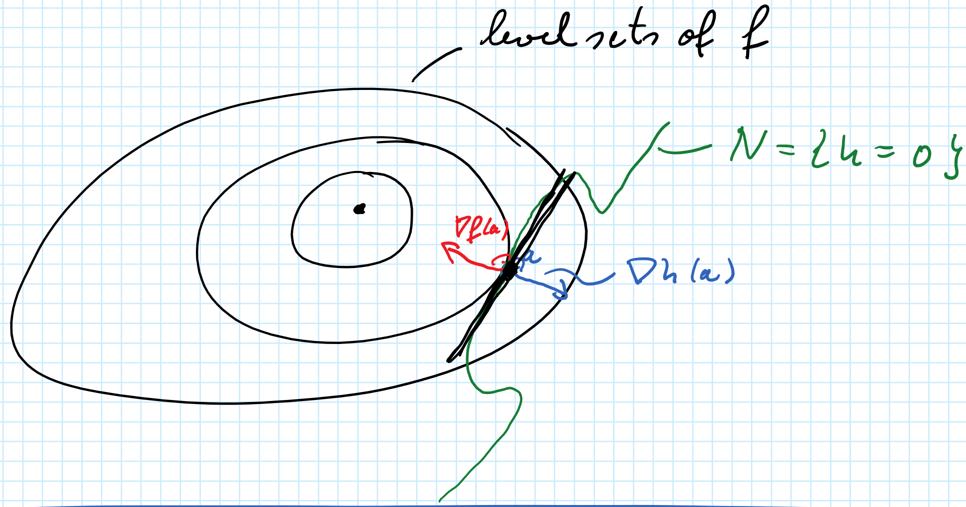


<u>Thm.:</u> Let $G \subset \mathbb{R}^n$ be open, $f, h \in C^2(G, \mathbb{R})$.
Let for $a \in N$ the necessary cond. $(*)$ be satisfied,
i.e. there exists $\lambda \in \mathbb{R}$ s.t. $\nabla F_\lambda(a) := \nabla(f - \lambda h)(a) = 0$.

(a) If $D^2 F|_a (v, v) > 0$ for all $v \in \mathbb{R}^n \setminus \{0\}$

$Dh|_a v = 0$, then f has a strict local minimum at a under the constraint $h = 0$.

(b) + (c) analogously.

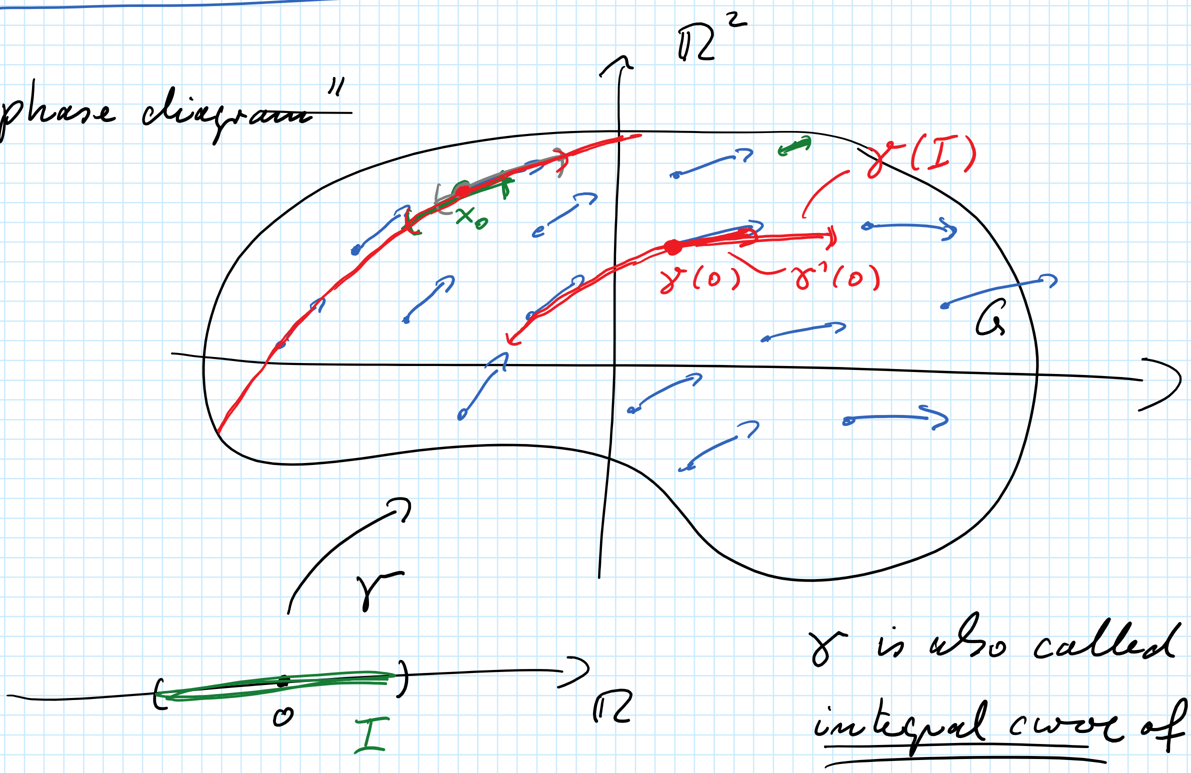
<u>Ordinary differential equations</u>

<u>Def.:</u> Let $G \subset \mathbb{R}^n$ open, $v \in C(G, \mathbb{R}^n)$ a continuous vector field, and $I \subset \mathbb{R}$ an open interval containing $0 \in \mathbb{R}$. A curve $\gamma \in C^1(I, G)$ is a <u>solution</u> of the <u>autonomous first order ODE</u>

$\boxed{\gamma' = v(\gamma)}$ with <u>initial datum</u> $x_0 \in G$

iff $\gamma' = v \circ \gamma$, i.e. $\gamma'(t) = v(\gamma(t))\ \forall t \in I$, iff $\gamma' = v \circ \gamma$, i.e. $\gamma'(t) = v(\gamma(t)) \ \forall t \in I$, and $\gamma(0) = x_0$.

„phase diagram"



γ is also called an <u>integral curve</u> of v

<u>Ex. 3.:</u> Determine and draw some integral curves for the vector fields $v: \mathbb{R}^2 \to \mathbb{R}^2$, $(x,y) \mapsto v(x,y) = \begin{pmatrix} -y \\ x \end{pmatrix}$

$w: \mathbb{R}^2 \to \mathbb{R}^2$, $(x,y) \mapsto w(x,y) = \begin{pmatrix} x \\ y \end{pmatrix}$

<u>Def.:</u> Let $m \in \mathbb{N}$. An <u>autonomous mth order ODE</u> on a domain $D \subset \mathbb{R}^n$ is given by a cont. fct.

$$f: D \times \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{(m-1) \text{ copies}} \to \mathbb{R}^n$$

and the equation

$$\gamma^{(m)} = f(\gamma, \gamma', \gamma'', \ldots, \gamma^{(m-1)}).$$

$0 \quad = f(0,0,0,\dots,0 \quad).$

Now $\gamma \in C^m(I,D)$ is called a solution with initial datum $(x_0, y_1, \dots, y_{m-1})$, iff

$$\gamma^{(m)}(t) = f(\gamma(t), \gamma'(t), \dots, \gamma^{(m-1)}(t)) \quad \forall t \in I$$

and $\gamma(0) = x_0$ and $\gamma^{(j)}(0) = y_j \quad \forall j = 1, \dots, m-1.$

Def.: Let $J \subset \mathbb{R}$ be an open interval. A cont. map

$$v : J \times D \to \mathbb{R}^n, \quad (t,x) \mapsto v(t,x)$$

is called a time-dependent vector field. The ODE

$$\boxed{\gamma' = v(t,\gamma)}$$

is called a non-autonomous first order ODE.

If $I \subset J$ is an open subinterval, $t_0 \in I$, $x_0 \in D$, then $\gamma : I \to D$ is a solution with initial value x_0 for initial time t_0, iff

$$\gamma'(t) = v(t, \gamma(t)) \quad \forall t \in I$$

and $\gamma(t_0) = x_0$.

All the above types of ODEs can be reduced to autonomous first order ODEs. !

Def.: Let $U \subset \mathbb{R} \times \mathbb{R}^n$ and $v \in C(U, \mathbb{R}^n)$.

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(a) We say that v satisfies a <u>Lipschitz condition</u>, iff there exist $L \geq 0$ such that

$$\forall (t,x), (t,y) \in U : \| v(t,x) - v(t,y) \| \leq L \| x - y \|.$$

(b) We say that v satisfies a <u>local Lip. cond.</u>, iff every $(t,x) \in U$ admits a neighbourhood $V \subset U$ such that $v|_V$ satisfies a Lip. cond.

<u>Thm. Picard-Lindelöf</u>

Let $U \subset \mathbb{R} \times \mathbb{R}^n$ be a domain and let $v \in C(U, \mathbb{R}^n)$ satisfy a local Lip. cond.

<u>Local existence:</u> For any $(t_0, x_0) \in U$ there exists $\delta > 0$ and a curve $\gamma \in C^1((t_0 - \delta, t_0 + \delta), \mathbb{R}^n)$ that is a solution of $\gamma' = v(t, \gamma)$ with initial datum $\gamma(t_0) = x_0$.

<u>Uniqueness:</u> If $J \subset \mathbb{R}$ is an interval with $t_0 \in J$ and $\tilde{\gamma} : J \to \mathbb{R}^n$ solves $\gamma' = v(t, \gamma)$ with $\tilde{\gamma}(t_0) = x_0$, then $\tilde{\gamma}(t) = \gamma(t) \quad \forall t \in J \cap (t_0 - \delta, t_0 + \delta)$.

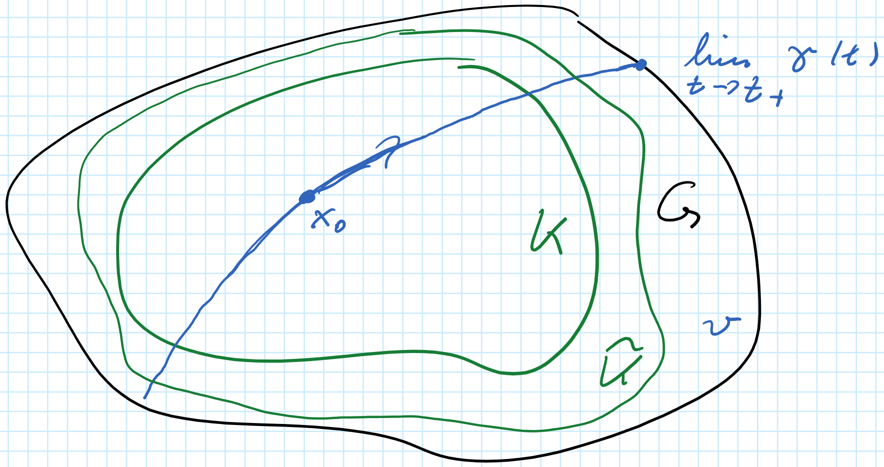
<u>Def.:</u> Let $v \in C(J \times G, \mathbb{R}^n)$ satisfy a loc. L.c.

__Def.:__ Let $v \in C(\mathcal{J} \times G, \mathbb{R}^n)$ satisfy a loc. L.c.
A solution $\gamma : I \to G$ of $\gamma' = v(t,\gamma)$ is called a __maximal solution__, iff the following holds:
If $I \subset \tilde{I} \subset \mathcal{J}$ and $\tilde{\gamma} : \tilde{I} \to G$ is a sol. of $\gamma' = v(t,\gamma)$ with $\tilde{\gamma}|_I = \gamma$, then $\tilde{I} = I$.

__Corollary:__ Under the cond. of P.L. there exists for any initial value a unique maximal solution.

__Thm.:__ Let $\underset{=\mathbb{R}}{\mathcal{J}} = (j_-, j_+) \subset \mathbb{R}$, $G \subset \mathbb{R}^n$ a domain, and $v \in C(\mathcal{J} \times G, \mathbb{R}^n)$ satisfy a loc. Lip. cond.
Let $\gamma : (t_-(t_0,x_0), t_+(t_0,x_0)) \to G$ be the maximal solution of $\gamma' = v(t,\gamma)$ for the initial value $(t_0, x_0) \in \mathcal{J} \times G$.
If $t_+(t_0,x_0) < j_+$, then for any compact $K \subset G$ there exists $0 < \tau_K < t_+(t_0,x_0)$ such that

$$\gamma(t) \notin K \quad \text{for all } t \in (\tau_K, t_+(t_0,x_0)).$$



<u>Def.:</u> A vector field $v \in C^1(G, \mathbb{R}^n)$ is <u>complete</u>, iff there exists a global solution $\gamma_{x_0} \in C^1(\mathbb{R}, G)$ of $\gamma' = v(\gamma)$ with $\gamma_{x_0}(0) = x_0$ for any initial value $x_0 \in G$.

The associated <u>flow</u> is

$$\Phi : \mathbb{R} \times G \to G, \quad (t, x) \mapsto \Phi(t,x) := \gamma_x(t)$$

and

$$\Phi_t : G \to G, \quad x \mapsto \Phi_t(x) := \Phi(t, x)$$

is called the <u>flow maps</u> at time t. It satisfies

$$\Phi_t \circ \Phi_s = \Phi_{t+s} \qquad \forall t, s \in \mathbb{R}$$

i.e. $\mathbb{R} \to \mathrm{Bij}(G \to G)$, $t \mapsto \Phi_t$ is a group action of $(\mathbb{R}, +)$ on the set G.

<u>Thm.:</u> If v satisfies a loc. Lip. cond., then the corresp. flow maps $\Phi_t : G \to G$ are continuous.

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If $v \in C^1$, then the flow maps $\Phi_t : G \to G$ are also C^1.