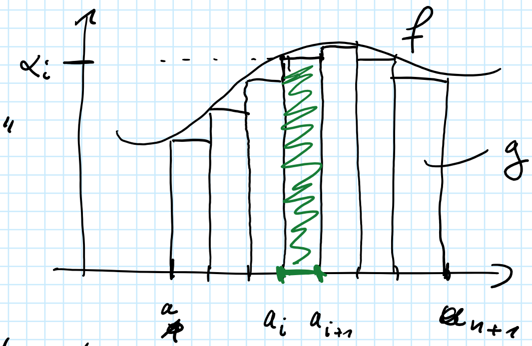
Motivation:

a) **Idea of the Riemann integral:**

Approximate f by „stair fcts"

$$g(x) = \sum_{i=1}^{n} \alpha_i \, \chi_{[a_i, a_{i+1}]}(x)$$



where for $A \subset \mathbb{R}$ the characteristic function of A is

$$\chi_A(x) := \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

Principle:
Decompose the domain into intervals (rectangles, cubes, ...)

The integral of a stair fct. is

$$\int g(x)\,dx := \sum_{i=1}^{n} \alpha_i (a_{i+1} - a_i)$$

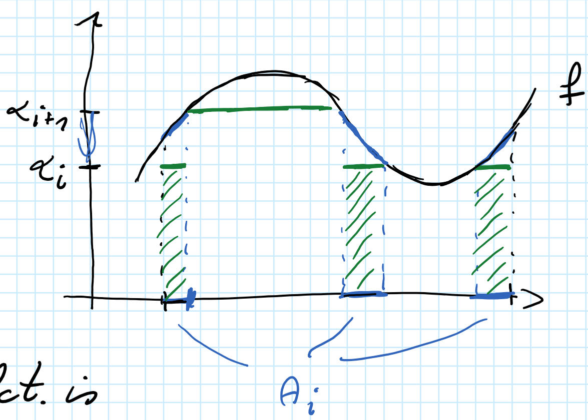
b) **Idea of the Lebesgue integral:**

Decompose the range of the fcts. into intervals $[\alpha_i, \alpha_{i+1})$ and approximate by „simple fcts."

$$g(x) := \sum_{i=1}^{n} \alpha_i \, \chi_{A_i}(x)$$

e.g. $A_i = f^{-1}([\alpha_i, \alpha_{i+1}))$

(not intervals in general)



The integral of a simple fct. is

$$\int g(x)\,dx := \sum_{i=1}^{n} \alpha_i \, \lambda(A_i)$$

where $\lambda(A_i)$ is the „length" of A_i (area, volume, measure)

<u>Example:</u> $f(x) = \chi_{\mathbb{Q}\cap[0,1]}(x)$ is <u>not</u> Riemann integrable, but it is Lebesgue integrable:

$$\int_0^1 f(x)\,dx = 1 \cdot \underbrace{\lambda(\mathbb{Q}\cap[0,1])}_{=0} + 0 \cdot \underbrace{\lambda([0,1]\setminus\mathbb{Q})}_{=1} = 0$$

$\Rightarrow$ Two advantages of Leb. int.:

(1) There are more integrable fcts. $\Rightarrow$
Spaces of Lebesgue integrable fcts are complete.

(2) The Lebesgue integral can be defined on all spaces where one can define a measure λ.
(not only on $\mathbb{R}$ or $\mathbb{R}^n$).

<u>Basic notions of measure theory:</u>

<u>Def.:</u> A family $\mathcal{A} \subset \mathcal{P}(X)$ of subsets of a set X is a <u>σ-algebra</u>, iff

countable

(i) $\emptyset \in \mathcal{A}$

(ii) $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$

(iii) $A_n \in \mathcal{A}$ for $n \in \mathbb{N} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$

The elements of $\mathcal{A}$ are called the $\mathcal{A}$-<u>measurable sets</u>.

<u>Prop.:</u> Let $\mathcal{A}$ be a σ-algebra on X. Then

(a) $X \in \mathcal{A}$ $\quad \left(\emptyset \in \mathcal{A} \overset{(ii)}{\Rightarrow} X = \emptyset^c \in \mathcal{A}\right)$

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(a) $X \in \mathcal{A}$ $\quad (\emptyset \in \mathcal{A} \overset{(ii)}{\Longrightarrow} X = \emptyset^c \in \mathcal{A})$

(b) $A_k \in \mathcal{A}$ for $k \in \mathbb{N} \Rightarrow \bigcap_{k=1}^{\infty} A_k \in \mathcal{A}$

(c) $A, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A}$, $A \cap B \in \mathcal{A}$, and $A \setminus B \in \mathcal{A}$.

Exc. 1: Prove this!

Sol. 1: (b) $A_k \in \mathcal{A} \;\forall k \in \mathbb{N} \overset{(ii)}{\Longrightarrow} A_k^c \in \mathcal{A} \;\forall k \in \mathbb{N}$

$$\overset{(iii)}{\Longrightarrow} \bigcup_{k \in \mathbb{N}} A_k^c \in \mathcal{A}$$

$$\overset{(ii)}{\Longrightarrow} \bigcap A_k = \left(\bigcup A_k^c\right)^c \in \mathcal{A}$$

(c) Let $A, B \in \mathcal{A}$ and put $A_1 = A$, $A_2 = B$, $A_k = \emptyset \;\forall k \geq 3$

$\Rightarrow A \cup B = \bigcup_{k \in \mathbb{N}} A_k \overset{(iii)}{\in} \mathcal{A}$ $\qquad$ $A_k = X$ for $A \cap B \in \mathcal{A}$

$A \setminus B = A \cap B^c \in \mathcal{A}$ $\qquad \boxtimes$

Examples: • $\mathcal{P}(X)$ and $\{\emptyset, X\}$ are σ-alg. on X.

• If $\mathcal{A}_j$, $j \in I$, are σ-alg. on X, so is $\bigcap_{j \in I} \mathcal{A}_j$ //

Def.: Let $\mathcal{F} \subset \mathcal{P}(X)$. Then the σ-alg. generated by $\mathcal{F}$ is

$$\mathcal{A}_{\mathcal{F}} := \bigcap_{\substack{\mathcal{B} \text{ is } \sigma\text{-alg.} \\ \mathcal{F} \subset \mathcal{B}}} \mathcal{B}$$

Any $\mathcal{F} \subset \mathcal{P}(X)$ that generates $\mathcal{A}$ is called a generating system for $\mathcal{A}$.

Def.: Let $(X, \mathcal{T})$ be a topological space. Then

Def.: Let $(X, \mathcal{T})$ be a topological space. Then $\mathcal{A}_{\mathcal{T}} =: \mathcal{B}$ is called the Borel-σ-algebra on X.

Def.: Let $\mathcal{A} \subset \mathcal{P}(X)$ be a σ-alg.

A map $\mu : \mathcal{A} \to [0, \infty]$ is called a measure, iff

(i) $\mu(\emptyset) = 0$

(ii) For pairwise disjoint sets $A_n \in \mathcal{A}$, $n \in \mathbb{N}$,

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n) \qquad (\sigma\text{-additivity})$$

If $\mu(X) < \infty$, then μ is called a finite measure.

If $X = \bigcup_{n=1}^{\infty} A_n$ with $\mu(A_n) < \infty \ \forall n$, then μ is called σ-finite.

The pair $(X, \mathcal{A})$ is called a measurable space, the triple $(X, \mathcal{A}, \mu)$ is called a measure space.

Example: Let X be a set and $x_0 \in X$. Then

$$\nu : \mathcal{P}(X) \to [0, \infty], \ A \mapsto \nu(A) := \begin{cases} |A| & \text{if } A \text{ is finite} \\ \infty & \text{otherwise} \end{cases}$$

„counting measure"

and

$$\delta_{x_0} : \mathcal{P}(X) \to [0, \infty], \ A \mapsto \delta_{x_0}(A) := \begin{cases} 1 & \text{if } x_0 \in A \\ 0 & \text{otherwise} \end{cases}$$

„Dirac measure at x_0"

are measures.

Prop.: Let μ be a measure on $(X, \mathcal{A})$ and $A, B \in \mathcal{A}$.

Th

Prop.: Let μ be a measure on $(X, \mathcal{A})$ and $A, B \in \mathcal{A}$.

Then
$$\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B)$$

and if $A \subset B$

$$\mu(B) = \mu(A) + \mu(B \setminus A) \quad \Rightarrow \quad \boxed{\mu(A) \leq \mu(B)}$$

monotonicity

For $A_j \in \mathcal{A}$, $j \in \mathbb{N}$,
$$\mu\left(\bigcup_{j=1}^{\infty} A_j\right) \leq \sum_{j=1}^{\infty} \mu(A_j) \qquad \text{(sub-additivity)}$$

and if $A_j \subset A_{j+1}$ $\forall j$, then
$$\lim_{j \to \infty} \mu(A_j) = \mu\left(\bigcup_{j=1}^{\infty} A_j\right)$$

Def.: Let $(X, \mathcal{A})$ and $(Y, \mathcal{C})$ be measure spaces.
A map $f : X \to Y$ is called $\mathcal{A}$-$\mathcal{C}$-measurable,
iff
$$C \in \mathcal{C} \Rightarrow f^{-1}(C) \in \mathcal{A}.$$

If μ is a measure on $(X, \mathcal{A})$ then the measure
$$f^*\mu : \mathcal{C} \to [0, \infty], \; C \mapsto f^*\mu(C) := \mu(f^{-1}(C))$$
is called its push-forward under f.

Remark: Terminology from probability theory

A measure space $(X, \mathcal{A}, \mu)$ with $\mu(X) = 1$ is called a probability space. Then the elements ~~of~~ $A \in \mathcal{A}$ are called events and $\mu(A)$ the probability of the called a probability space. Then the elements ~~of~~ of $\mathcal{A}$ are called events and $\mu(A)$ the probability of the event. Measurable fcts $f: X \to Y$, $(Y, \mathcal{C})$ a measurable space, are called random variables and the probability measure $f_*\mu$ is called the distribution of f.

Example: Throwing a dice once

$X_1 = \{1,2,3,4,5,6\}$, $\mu(X) = 1$

$\mu(\{1\}) = \frac{1}{6} = \mu(\{2\}) = \mu(\{3\}) \cdots = \mu(\{6\})$

$\mathcal{A} = \mathcal{P}(X) = \{\emptyset, \{1\}, \ldots, \{2\}, \underbrace{\{1,3,5\}}, \{2,4,6\}, \ldots\}$

$|\mathcal{P}(X)| = 2^6$

event that the outcome is odd

$\mu(\{1,3,5\}) = \frac{3}{6} = \frac{1}{2}$

throwing a dice twice: $X_2 = X_1 \times X_1$, $\mu_2(\{(a,b)\}) := \mu_1(\{a\}) \cdot \mu_2(\{b\}) = \frac{1}{36}$.

$f: X_2 \to Y$, $(a,b) \mapsto a+b \in \{2, \ldots, 12\} =: Y$

$f_*\mu_2(\{3\}) := \mu_2(f^{-1}(\{3\})) = \mu_2(\{(1,2),(2,1)\}) = \frac{2}{36} = \frac{1}{18}$

Thm.: There is a unique measure λ on $(\mathbb{R}^n, \mathcal{B})$ that is translation invariant (i.e. $\lambda(A+x) = \lambda(A)$ $\forall A \in \mathcal{B}\ \forall x \in \mathbb{R}^n$) and normalised to $\lambda((0,1)^n) = 1$. It is called the Lebesgue-Borel measure and its completion is called the Lebesgue measure.

Exercise 2: Show that $\lambda(\mathbb{Q}) = 0$.

Sol. 2: We show that for all $x \in \mathbb{R}$ $\lambda(\{x\}) = 0$.

Then $\lambda(\mathbb{Q}) = 0$ follows, since $\mathbb{Q} = \bigcup_{r=1}^{\infty} \{q_r\}$.

By transl. invariance $\lambda(\{x\}) = \lambda(\{y\})$ $\forall x, y \in \mathbb{R}$

Let $\lambda(\{x\}) = \varepsilon \geq 0$. $\{\frac{1}{n} \mid n \in \mathbb{N}\} = N = \bigcup_{n \in \mathbb{N}} \{\frac{1}{n}\}$

$$\lambda(N) \overset{(iii)}{=} \sum_{n=1}^{\infty} \lambda(\{\tfrac{1}{n}\}) = \sum_{n=1}^{\infty} \varepsilon \overset{\text{monot.}}{\leq} \lambda([0,1]) = 1.$$

$\Rightarrow \varepsilon = 0.$ $\qquad \mathbb{C} \cong \mathbb{R}^2$

Note that no translation invariant and normalised measure exists on $\mathcal{P}(\mathbb{R})$!

Basic notions of integration theory

Def.: A fct. $g: X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$ is called simple, if $g(X) =: \{\alpha_1, .., \alpha_r\}$ is finite, i.e.

$$g(x) = \sum_{j=1}^{r} \alpha_j \chi_{A_j}(x) \quad \text{with } A_j \cap A_i = \emptyset \text{ for } i \neq j.$$

$$g = \sum_{j=1}^{r} \alpha_j \chi_{A_j}$$

Def.: Let $(X, \mathcal{A}, \mu)$ be a measure space and $g : X \to [0,\infty]$ a simple and measurable, then

$$\boxed{\int_X g \, d\mu} := \sum_{j=1}^{r} \alpha_j \, \mu(A_j)$$

For a measurable fct. $f : X \to [0,\infty]$

$$\boxed{\int_X f \, d\mu} = \sup \left\{ \int_X g \, d\mu \;\middle|\; g : X \to [0,\infty] \text{ simple, measurable and } g \leq f \right\}$$

$\in \mathbb{R}$

Prop.: Let $f, g : X \to [0,\infty]$ be measurable and $\alpha \geq 0$.

Then (also hold for integrable fcts.)

(i) $\int \alpha f \, d\mu = \alpha \int f \, d\mu$

(ii) $\int (f+g) \, d\mu = \int f \, d\mu + \int g \, d\mu$

(i), (ii): $\int$ is linear

(iii) $f \leq g \Rightarrow \int f \, d\mu \leq \int g \, d\mu$ (monotone)

Thm.: (monotone convergence)

Let $f_n : X \to [0,\infty]$ measurable and $f_n \leq f_{n+1}$ for ~~all~~ $n \in \mathbb{N}$. Let $f := \lim_{n\to\infty} f_n$ (pointwise), then

$$\lim_{n\to\infty} \int f_n \, d\mu = \int f \, d\mu$$

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Corollary: Fatou's lemma

Let $f_n : X \to [0,\infty]$ be measurable. Then

$$\int \liminf_{n\to\infty} f_n \, d\mu \leq \liminf_{n\to\infty} \int f_n \, d\mu$$

Def.: A measurable fct. $f : X \to \overline{\mathbb{R}}$ is integrable, iff for $f_+ := \max(f, 0)$ and $f_- := \max(-f, 0)$ it holds that $\boxed{\int f_+ \, d\mu < \infty}$ and $\boxed{\int f_- \, d\mu < \infty}$.

$f = f_+ - f_-$

Then $\boxed{\int f \, d\mu := \int f_+ \, d\mu - \int f_- \, d\mu.}$

Def.: We say that a property of a point $x \in X$ holds almost surely or almost everywhere w.r.t. a measure μ on X, if it holds for $x \in A \subset X$ and

$$\mu(X \setminus A) = 0,$$

i.e. if it fails to hold a null set only.

Examples:

- A real number is almost surely irrational w.r.t. Lebesgue measure on $\mathbb{R}$.

$$\int_{\mathbb{R}} \chi_{\mathbb{Q}}(x) \, d\lambda = 0$$

- Let $f : X \to [0,\infty]$ be measurable. Then

$$\int_X f \, d\mu = 0 \iff f = 0 \text{ almost everywhere.}$$

$\Rightarrow$ changing an integrable fct. f on a null set does not change $\int f \, d\mu$. $\Rightarrow$ for integrable fcts. we do not include $\pm\infty$ in the range anymore.

Rem.: • Every Riemann integrable fct. $f : [a,b] \to \mathbb{R}$ is also Lebesgue integrable and the integrals coincide.

• A fct. $f : X \to \mathbb{C}$ is integrable, iff $|f|$ is integrable and
$$\int f \, d\mu := \int \operatorname{Re} f \, d\mu + i \int \operatorname{Im} f \, d\mu .$$

• Analogously for $f : X \to W$ (W - finite dim.)

• For $f : X \to W$, W a Banach space, the generalisation is called the Bochner integral.

Def.: Let $(X, \mathcal{A}, \mu)$ be a measure space and $1 \le p < \infty$. Then
$$\mathcal{L}^p(X, \mu) := \{ f : X \to \mathbb{C} \mid f \text{ is measurable and } |f|^p \text{ integrable} \}$$
and for $f \in \mathcal{L}^p(X, \mu)$
$$\|f\|_{L^p} := \left(\int |f|^p \, d\mu \right)^{1/p} < \infty .$$

Moreover, $L^p(X, \mu) := \mathcal{L}^p(X, \mu)/\sim$ w.r.t. the equivalence relation
$$f \sim g :\iff f = g \text{ almost everywhere.}$$

equivalence relation

$$f \sim g \;:\Leftrightarrow\; f = g \quad \text{almost everywhere.}$$

<u>Thm.:</u> Let $(X, \mathcal{A}, \mu)$ be a measure space and $1 \leq p \leq \infty$. Then $(L^p(X,\mu), \|\cdot\|_{L^p})$ is a Banach space.

<u>Thm.:</u> <u>(dominated convergence)</u>

Let $f_n : X \to \mathbb{C}$ be (p-integrable) measurable, $n \in \mathbb{N}$, and assume that $f(x) := \lim_{n\to\infty} f_n(x)$ exists for almost all $x \in X$.

If for some $g \in L^p(X,\mu)$, $1 \leq p < \infty$ it holds that $|f_n| \leq |g|$ almost everywhere and for all $n \in \mathbb{N}$, then $f_n, f \in L^p(X,\mu)$ and

$$\lim_{n\to\infty} \|f_n - f\|_{L^p} = 0$$

$p=1$

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu$$

i.e. $f_n \to f$ in $L^p(X,\mu)$.

<u>Definition</u>: Let $(X, \mathcal{A}, \mu)$ be a measure space.

$|f| : X \to [0,\infty]$

For measurable $f : X \to \mathbb{C}$ we define

$$\|f\|_{L^\infty} := \inf\{\, 0 \leq \lambda \leq \infty \mid \mu\left(|f|^{-1}((\lambda,\infty])\right) = 0 \,\}$$

$$=: \operatorname{ess\,sup} |f|.$$

e.g. $\|\chi_{\mathbb{Q}}\|_{L^\infty} = 0$

$$\mathcal{L}^\infty(X) := \{\, f : X \to \mathbb{C} \mid f \text{ measurable and } \|f\|_{L^\infty} < \infty \,\}$$

and $L^\infty(X) := \mathcal{L}^\infty(X)/\sim$

and $L^\infty(X) = \mathcal{L}^\infty(X)/\sim$.

<u>Examples:</u> (a) If $\mu(X) < \infty$ and $f \in L^\infty(X)$, then

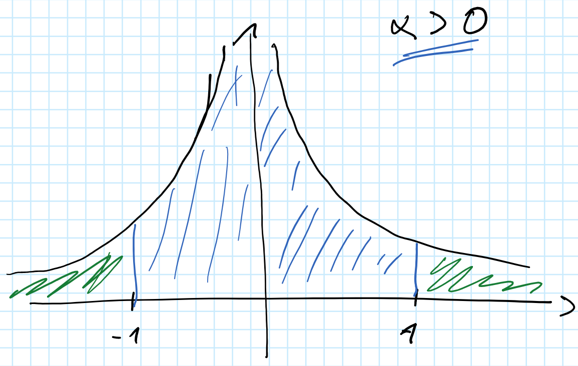
$$\int_X |f| \, d\mu \leq \int_X \underbrace{\|f\|_{L^\infty}}_{\|f\|_\infty} d\mu = \|f\|_{L^\infty} \cdot \mu(X)$$

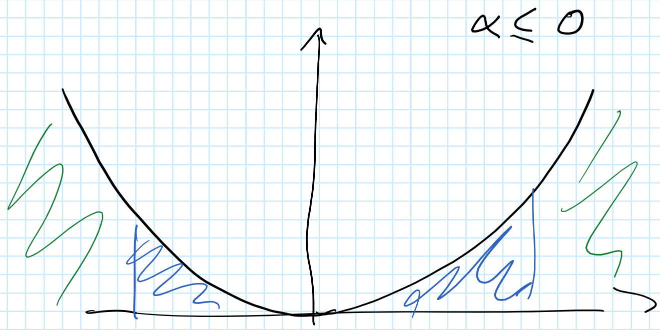
In part. $L^\infty(X) \subset L^1(X)$ in this case

Actually, $L^p(X) \subset L^q(X)$ if $p > q$ and $\mu(X) < \infty$.

(b) $X = \mathbb{R}^n$, $\mu = \lambda^n$, then $\mu(X) = \infty$ and (a) does not apply. But for $A \subset \mathbb{R}^n$ with $\lambda(A) < \infty$, then (a) does apply: $L^p(A, \lambda) \subset L^q(A, \lambda)$ if $p > q$.

Let $f: \mathbb{R}^n \to \mathbb{R}$, $x \mapsto f(x) = \dfrac{1}{\|x\|^\alpha}$, $\alpha \in \mathbb{R}$





(i) $A = B_1(0) \subset \mathbb{R}^n$. Then for $\alpha > 0$ $f \notin L^\infty(A, \lambda)$

$$\int_{B_1(0)} |f|^p \, d\lambda = \int_{B_1(0)} \frac{1}{\|x\|^{\alpha \cdot p}} \, d\lambda = C_n \int_0^1 \frac{1}{r^{\alpha \cdot p}} r^{n-1} \, dr$$

$$= C_n \int_0^1 \frac{1}{r^{\alpha \cdot p + 1 - n}} \, dr$$

$$= \begin{cases} < \infty & \text{iff } \alpha p + 1 - n < 1 \\ & \alpha < \frac{n}{p} \\ = \infty & \text{iff } \alpha \geq \frac{n}{p} \end{cases}$$

$$\Rightarrow L^p(B_1(0)) \subsetneq L^q(B_1(0)) \quad \text{if } p > q$$

(ii) $\Omega = \mathbb{R}^n \setminus B_1(0)$. Then

$$\int_\Omega |f|^p \, d\Omega = c_n \int_1^\infty \frac{1}{r^{\alpha p + 1 - n}} \, dr$$

$$= \begin{cases} < \infty & \alpha > \frac{n}{p} \\ = \infty & \alpha \geq \frac{n}{p} \end{cases}$$

$$\Rightarrow L^p(\mathbb{R}^n) \subsetneq L^q(\mathbb{R}^n) \text{ and}$$

$$L^q(\mathbb{R}^n) \subsetneq L^p(\mathbb{R}^n)$$

for $p \neq q$

c) $f_n \quad \mathbb{R} \to \mathbb{R}, \quad x \mapsto \chi_{[n,n+1]}(x)$

Then $f_n \in L^p \quad \forall p \qquad (\|f_n\|_p = 1)$

and $f_n \xrightarrow{p.w.} f \equiv 0 \in L^p \quad \forall p$

but $\|f_n - f\|_{L^p} = \|f_n\|_{L^p} = 1 \qquad \forall n$

$\not\to 0$

„no dominating fct. g exists" $\quad g = \chi_{[0,\infty)}$

Thm: Hölder inequality

Let $f, g : X \to \mathbb{C}$ be measurable and $1 \leq p, q \leq \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$ where $\frac{1}{\infty} := 0$. Then

$$\boxed{\|fg\|_{L^1} \leq \|f\|_{L^p} \cdot \|g\|_{L^q}}$$

(conjugate exponents
$p = q = 2$
$p = \infty, q = 1$

For $p = q = 2$ this is the Cauchy-Schwarz inequal. on the Hilbert space L^2

$f, g \in L^2 \Rightarrow \bar{f} g \in L^1$

$$\left| \int \bar{f} g \, d\mu \right| \leq \int |\bar{f} g| \, d\mu \overset{\text{Hölder}}{\leq} \|f\|_{L^2} \cdot \|g\|_{L^2}$$

$$\shortparallel$$

$$|\langle f, g \rangle_{L^2}| \leq \|f\|_{L^2} \cdot \|g\|_{L^2}$$

Thm: Minkowski inequality (Δ-inequ. for p-norms)

Thm: Minkowski inequality (Δ-inequ. for p-norms)

Let $f, g : X \to [0, \infty]$ be measurable and $1 \leq p \leq \infty$.

Then $\quad \|f+g\|_p \leq \|f\|_p + \|g\|_p$