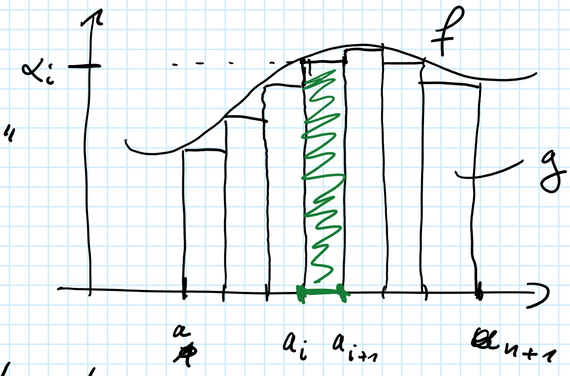
Motivation:

a) Idea of the Riemann integral:

Approximate f by „stair fcts"

$$g(x) = \sum_{i=1}^{n} \alpha_i \chi_{[a_i, a_{i+1}]}(x)$$



where for $A \subset \mathbb{R}$ the characteristic function of A is

$$\chi_A(x) := \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

Principle:
Decompose the domain into intervals (rectangles, cubes, ...)

The integral of a stair fct. is

$$\int g(x)\,dx := \sum_{i=1}^{n} \alpha_i (a_{i+1} - a_i)$$

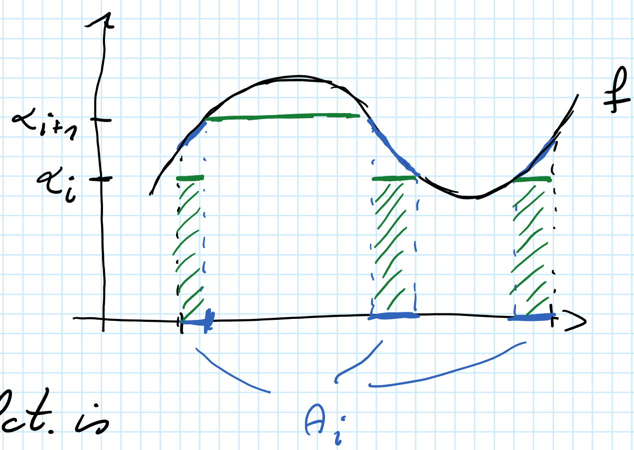
b) Idea of the Lebesgue integral:

Decompose the range of the fcts. into intervals $[\alpha_i, \alpha_{i+1})$ and approximate by „simple fcts."

$$g(x) := \sum_{i=1}^{n} \alpha_i \chi_{A_i}(x)$$

e.g. $A_i = f^{-1}([\alpha_i, \alpha_{i+1}))$

(not intervals in general)



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$$\int g(x)\,dx := \sum_{i=1}^{n} \alpha_i \,\lambda(A_i)$$

where $\lambda(A_i)$ is the „length" of A_i (area, volume, measure)

<u>Example:</u> $f(x) = \chi_{\mathbb{Q}\cap[0,1]}(x)$ is <u>not</u> Riemann integrable, but it is Lebesgue integrable:

$$\int_0^1 f(x)\,dx = 1\cdot \underbrace{\lambda(\mathbb{Q}\cap[0,1])}_{=0} + 0\cdot \underbrace{\lambda([0,1]\setminus\mathbb{Q})}_{=1} = 0$$

$\Rightarrow$ Two advantages of Leb. int.:

(1) There are more integrable fcts. $\Rightarrow$
Spaces of Lebesgue integrable fcts are complete.

(2) The Lebesgue integral can be defined on all spaces where one can define a measure λ.
(not only on $\mathbb{R}$ or $\mathbb{R}^n$).

<u>Basic notions</u> of <u>measure theory</u>:

<u>Def.:</u> A family $\mathcal{A} \subset \mathcal{P}(X)$ of subsets of a set X is a <u>σ-algebra</u>, iff

(i) $\emptyset \in \mathcal{A}$

(ii) $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$

(iii) $A_k \in \mathcal{A}$ for $k \in \mathbb{N} \Rightarrow \bigcup_{k=1}^{\infty} A_k \in \mathcal{A}$

$$r=1$$

The elements of $\mathcal{A}$ are called the $\mathcal{A}$-measurable sets.

Prop.: Let $\mathcal{A}$ be a σ-algebra on X. Then

(a) $X \in \mathcal{A}$

(b) $A_r \in \mathcal{A}$ for $r \in \mathbb{N} \Rightarrow \bigcap_{r=1}^{\infty} A_r \in \mathcal{A}$

(c) $A, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A}$, $A \cap B \in \mathcal{A}$, and $A \setminus B \in \mathcal{A}$.

Exc. 1: Prove this!

Examples: • $\mathcal{P}(X)$ and $\{\emptyset, X\}$ are σ-alg. on X.

• If $\mathcal{A}_j$, $j \in I$, are σ-alg. on X, so is $\bigcap_{j \in I} \mathcal{A}_j$

Def.: Let $\mathcal{F} \subset \mathcal{P}(X)$. Then the σ-alg. generated by $\mathcal{F}$ is

$$\mathcal{A}_{\mathcal{F}} := \bigcap_{\substack{\mathcal{B} \text{ is } \sigma\text{-alg.} \\ \mathcal{F} \subset \mathcal{B}}} \mathcal{B}$$

Any $\mathcal{F} \subset \mathcal{P}(X)$ that generates $\mathcal{A}$ is called a generating system for $\mathcal{A}$.

Def.: Let $(X, \mathcal{T})$ be a topological space. Then

$\mathcal{A}_{\mathcal{T}} =: \mathcal{B}$ is called the Borel-σ-algebra on X.

Def.: Let $\mathcal{A} \subset \mathcal{P}(X)$ be a σ-alg.

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A map $\mu : \mathcal{A} \to [0,\infty]$ is called a measure, iff

(i) $\mu(\emptyset) = 0$

(ii) For pairwise disjoint sets $A_k \in \mathcal{A}$, $k \in \mathbb{N}$,

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k) \qquad (\sigma\text{-additivity})$$

If $\mu(X) < \infty$, then μ is called a finite measure.

If $X = \bigcup_{k=1}^{\infty} A_k$ with $\mu(A_k) < \infty \ \forall k$, then μ is called σ-finite.

The pair $(X, \mathcal{A})$ is called a measurable space, the triple $(X, \mathcal{A}, \mu)$ is called a measure space.

Example: Let X be a set and $x_0 \in X$. Then

$$\nu : \mathcal{P}(X) \to [0,\infty],\ A \mapsto \nu(A) := \begin{cases} |A| & \text{if } A \text{ is finite} \\ \infty & \text{otherwise} \end{cases}$$

„counting measure"

and

$$\delta_{x_0} : \mathcal{P}(X) \to [0,\infty],\ A \mapsto \delta_{x_0}(A) := \begin{cases} 1 & \text{if } x_0 \in A \\ 0 & \text{otherwise} \end{cases}$$

„Dirac measure at x_0"

are measures.

Prop.: Let μ be a measure on $(X, \mathcal{A})$ and $A, B \in \mathcal{A}$.

Then

$$\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B)$$

$$\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B)$$

and if $A \subset B$

$$\mu(B) = \mu(A) + \mu(B \setminus A) \quad \Rightarrow \quad \mu(A) \leq \mu(B)$$

For $A_j \in \mathcal{A}$, $j \in \mathbb{N}$,

$$\mu\Big(\bigcup_{j=1}^{\infty} A_j\Big) \leq \sum_{j=1}^{\infty} \mu(A_j) \qquad \text{(sub-additivity)}$$

and if $A_j \subset A_{j+1}$ $\forall j$, then

$$\lim_{j \to \infty} \mu(A_j) = \mu\Big(\bigcup_{j=1}^{\infty} A_j\Big)$$

<u>Def.</u>: Let $(X, \mathcal{A})$ and $(Y, \mathcal{C})$ be measure spaces. A map $f: X \to Y$ is called <u>$\mathcal{A}$-$\mathcal{C}$-measurable</u>, iff

$$C \in \mathcal{C} \Rightarrow f^{-1}(C) \in \mathcal{A}.$$

If μ is a measure on $(X, \mathcal{A})$ then the measure

$$f^*\mu : \mathcal{C} \to [0, \infty], \quad C \mapsto f^*\mu(C) := \mu(f^{-1}(C))$$

is called its <u>push-forward</u> under f.

<u>Remark: Terminology from probability theory</u>

A measure space $(X, \mathcal{A}, \mu)$ with $\mu(X) = 1$ is called a <u>probability space</u>. Then the elements ~~of~~ $A \in \mathcal{A}$

called a probability space. Then the elements $A \in \mathcal{A}$ are called events and $\mu(A)$ the probability of the event. Measurable fcts $f: X \to Y$, $(Y, \mathcal{C})$ a measurable space, are called random variables and the probability measure $f^*\mu$ is called the distribution of f.

Thm.: There is a unique measure λ on $(\mathbb{R}^n, \mathcal{B})$ that is translation invariant (i.e. $\lambda(A+x) = \lambda(A)$ $\forall A \in \mathcal{B}$ $\forall x \in \mathbb{R}^n$) and normalised to $\lambda((0,1)^n) = 1$. It is called the Lebesgue-Borel measure and its completion is called the Lebesgue measure.

Exercise 2: Show that $\lambda(\mathbb{Q}) = 0$.

Note that no translation invariant and normalised measure exists on $\mathcal{P}(\mathbb{R})$!

Basic notions of integration theory

Def.: A fct. $g: X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$ is called simple, if $g(X) =: \{\alpha_1, .., \alpha_n\}$ is finite, i.e.

$$g(x) = \sum_{j}^{n} \alpha_j \chi_{A_j}(x) \quad \text{with } A_j \cap A_i = \emptyset \text{ for } i \neq j.$$

$$g(x) = \sum_{j=1}^{r} \alpha_j \chi_{A_j}(x) \quad \text{with } A_j \cap A_i = \emptyset \text{ for } i \neq j.$$

Def.: Let $(X, \mathcal{A}, \mu)$ be a measure space and $g: X \to [0, \infty]$ a simple and measurable, then

$$\boxed{\int_X g \, d\mu} := \sum_{j=1}^{r} \alpha_j \mu(A_j)$$

For a measurable fct. $f: X \to [0, \infty]$

$$\boxed{\int_X f \, d\mu} = \sup\left\{ \int_X g \, d\mu \;\middle|\; g: X \to [0,\infty] \text{ simple, measurable and } g \leq f \right\}$$

Prop.: Let $f, g : X \to [0, \infty]$ be measurable and $\alpha \geq 0$.

Then (also hold for integrable fcts.)

(i) $\int \alpha f \, d\mu = \alpha \int f \, d\mu$

(ii) $\int (f+g) \, d\mu = \int f \, d\mu + \int g \, d\mu$

$\Big\}$ $\int$ is linear

(iii) $f \leq g \Rightarrow \int f \, d\mu \leq \int g \, d\mu$ (monotone)

Thm.: (monotone convergence)

Let $f_n : X \to [0, \infty]$ measurable and $f_n \leq f_{n+1}$ for ~~all~~ $n \in \mathbb{N}$. Let $f := \lim f_n$ (pointwise), then for ~~all~~ $n \in \mathbb{N}$. Let $f := \lim_{n\to\infty} f_n$ (pointwise), then

$$\lim_{n\to\infty} \int f_n \, d\mu = \int f \, d\mu$$

<u>Corollary</u>: <u>Fatou's lemma</u>

Let $f_n : X \to [0, \infty]$ be measurable. Then

$$\int \liminf_{n\to\infty} f_n \, d\mu \leq \liminf_{n\to\infty} \int f_n \, d\mu$$

<u>Def.</u>: A measurable fct. $f : X \to \overline{\mathbb{R}}$ is <u>integrable</u>, iff for $f_+ := \max(f, 0)$ and $f_- := \max(-f, 0)$ it holds that $\int f_+ \, d\mu < \infty$ and $\int f_- \, d\mu < \infty$.

Then $\boxed{\int f \, d\mu := \int f_+ \, d\mu - \int f_- \, d\mu.}$

<u>Def.</u>: We say that a property of a point $x \in X$ holds <u>almost surely</u> or <u>almost everywhere</u> w.r.t. a measure μ on X, if it holds for $x \in A \subset X$ and

$$\mu(X \setminus A) = 0,$$

i.e. if it fails to hold a <u>null set</u> only.

Examples: • A real number is almost surely irrational w.r.t. Lebesgue measure on $\mathbb{R}$.

• Let $f: X \to [0,\infty]$ be measurable. Then

$$\int_X f \, d\mu = 0 \iff f = 0 \text{ almost everywhere.}$$

$\Rightarrow$ changing an integrable fct. f on a null set does not change $\int f \, d\mu$. $\Rightarrow$ for integrable fcts. we do not include $\pm\infty$ in the range anymore.

Rem.: • Every Riemann integrable fct. $f: [a,b] \to \mathbb{R}$ is also Lebesgue integrable and the integrals coincide.

• A fct. $f: X \to \mathbb{C}$ is integrable, iff $|f|$ is integrable and

$$\int f \, d\mu := \int \operatorname{Re} f \, d\mu + i \int \operatorname{Im} f \, d\mu .$$

• Analogously for $f: X \to W$ (W-finite dim.)

• For $f: X \to W$, W a Banach space, the generalisation is called the Bochner integral.

Def.: Let $(X, \mathcal{A}, \mu)$ be a measure space and $1 \leq p < \infty$. Then

$$\mathcal{L}^p(X,\mu) := \{ f: X \to \mathbb{C} \mid f \text{ is measurable and } |f|^p \text{ integrable} \}$$

and for $f \in \mathcal{L}^p(X,\mu)$

and for $f \in \mathcal{L}^p(X, \mu)$

$$\|f\|_{L^p} := \left(\int |f|^p \, d\mu \right)^{1/p} < \infty .$$

Moreover, $L^p(X, \mu) := \mathcal{L}^p(X, \mu)/\sim$ w.r.t. the equivalence relation

$$f \sim g :\Leftrightarrow f = g \text{ almost everywhere.}$$

Thm.: Let $(X, \mathcal{A}, \mu)$ be a measure space and $1 \le p \le \infty$. Then $(L^p(X, \mu), \|\cdot\|_{L^p})$ is a Banach space.

Thm.: (dominated convergence)

Let $f_n : X \to \mathbb{C}$ be measurable, $n \in \mathbb{N}$, and assume that $f(x) := \lim_{n \to \infty} f_n(x)$ exists for almost all $x \in X$. If for some $g \in L^p(X, \mu)$, $1 \le p < \infty$ it holds that $|f_n| \le g$ almost everywhere and for all $n \in \mathbb{N}$, then $f_n, f \in L^p(X, \mu)$ and

$$\lim_{n \to \infty} \|f_n - f\|_{L^p} = 0$$

i.e. $f_n \to f$ in $L^p(X, \mu)$.