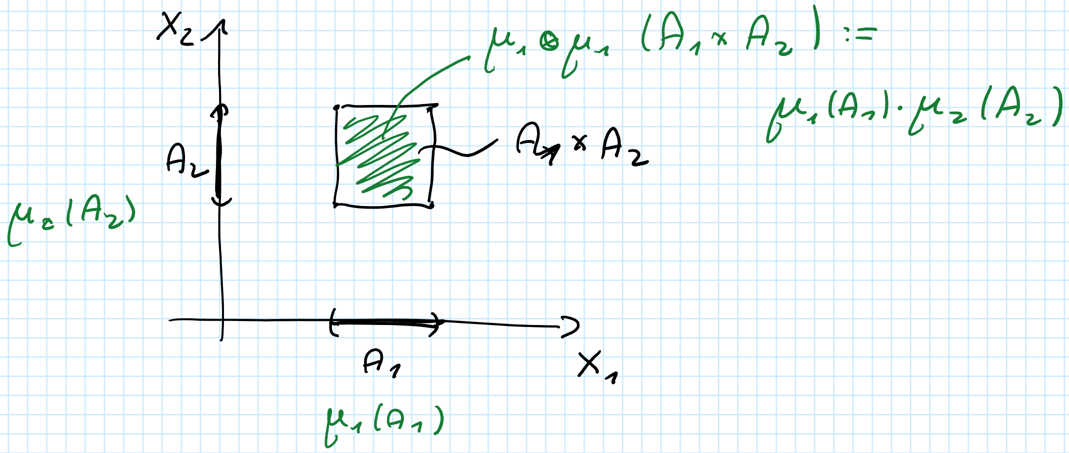
Addendum to integration theory:

Product measures and Fubini's theorem

$$\int\left(\int f(x,y)\,dx\right)dy \overset{?}{=} \int\left(\int f(x,y)\,dy\right)dx$$

Def.: Let $(X_1, \mathcal{A}_1)$ and $(X_2, \mathcal{A}_2)$ be measurable spaces. Then $\mathcal{A}_1 \otimes \mathcal{A}_2$ denotes the σ-algebra on $X_1 \times X_2$ generated by sets of the form $A_1 \times A_2 \subset X_1 \times X_1$ with $A_1 \in \mathcal{A}_1$ and $A_2 \in \mathcal{A}_2$, the so called product σ-alg.

Example: Let $\mathcal{B}^n \subset \mathcal{P}(\mathbb{R}^n)$ be the Borel-σ-algebra. Then $\mathcal{B}^n = \mathcal{B}^1 \otimes \cdots \otimes \mathcal{B}^1$.



Thm: Let $(X_1, \mathcal{A}_1, \mu_1)$ and $(X_2, \mathcal{A}_2, \mu_2)$ be σ-finite measure spaces. There exists a unique measure μ on $\mathcal{A}_1 \otimes \mathcal{A}_2$ such that for all $A_1 \in \mathcal{A}_1$, $A_2 \in \mathcal{A}_2$

$$\mu(A_1 \times A_2) = \mu_1(A_1) \cdot \mu_2(A_2),$$

called the product measure and denoted by $\mu = \mu_1 \otimes \mu_2$.

Example: The Lebesgue-Borel measure
$$\lambda^n = \lambda^1 \otimes \cdots \otimes \lambda^1 .$$

Theorem: Tonelli

Let $(X_1, \mathcal{A}_1, \mu_1)$ and $(X_2, \mathcal{A}_2, \mu_2)$ be σ-finite measure spaces. Let $f : X_1 \times X_2 \to [0,\infty]$ be $\mathcal{A}_1 \otimes \mathcal{A}_2$-measurable. Then

$$\int_{X_1 \times X_2} f \, d(\mu_1 \otimes \mu_2) = \int_{X_1} \underbrace{\left(\int_{X_2} f_{x_1} \, d\mu_2 \right)}_{\text{fct. of } x_1} d\mu_1$$

$$= \int_{X_2} \left(\int_{X_1} f_{x_2} \, d\mu_1 \right) d\mu_2$$

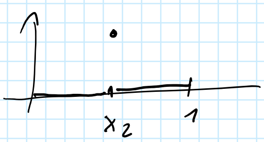
where $f_{x_1} : X_2 \to \overline{\mathbb{R}}$, $x_2 \mapsto f(x_1, x_2)$.

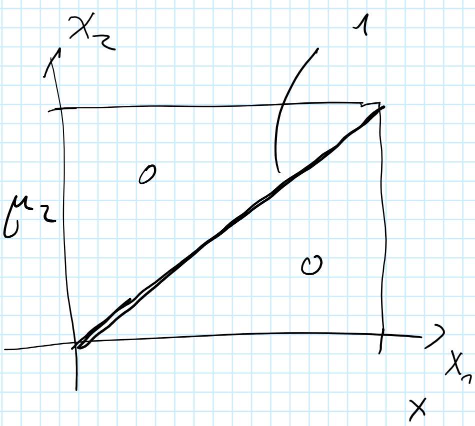
Ex.: $X_1 = X_2 = [0,1]$, $\mu_1 = \lambda^1$, $\mu_2 = \nu$ counting measure

(X_2, μ_2) is not σ-finite

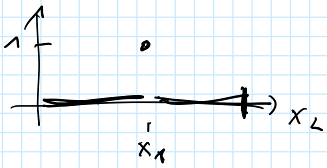
$$f : X_1 \times X_2 \to [0,\infty)$$
$$(x_1, x_2) \mapsto \delta_{x_1, x_2} := \begin{cases} 1 & \text{if } x_1 = x_2 \\ 0 & \text{otherwise} \end{cases}$$

$$\int \underbrace{f(x_1, x_2)}_{\substack{!! \\ f_{x_2}(x_1)}} d\mu_1 = 0 \quad , \quad \int \underbrace{\left(\int f_{x_2}(x_1) \, d\mu_1 \right)}_{0} d\mu_2 = 0$$





$$\underbrace{\int f(x_1,x_2)\,d\mu_2}_{f_{x_1}(x_2)} = 0\cdot \underbrace{\mu_2(X_{\{f=0\}})}_{=\infty} + 1\cdot \mu_2(\underbrace{X_{\{f=1\}}}_{\{x_1\}})$$



$$= 0 + 1 = 1$$

$$\int_0^1 \Big(\underbrace{\int f(x_1,x_2)\,d\mu_2}_{=1}\Big)\,d\mu_1 = 1$$

Thm.: Fubini

Let $(X_1, \mathcal{A}_1, \mu_1)$ and $(X_2, \mathcal{A}_2, \mu_2)$ be σ-fin. meas. spaces.

and $f: X_1 \times X_2 \to \mathbb{C}$ measurable.

(a) $\displaystyle\int_{X_1}\Big(\int_{X_2} |f_{x_1}|\,d\mu_2\Big)\,d\mu_1 < \infty$ or if $\displaystyle\int_{X_2}\Big(\int_{X_1} \ldots\, d\mu_1\Big)\,d\mu_2 < \infty$

iff $f \in L^1(X_1 \times X_2, \mu_1 \otimes \mu_2)$.

(b) If $f \in L^1(X_1 \times X_2, \mu_1 \otimes \mu_2)$, then

$$\int_{X_1\times X_2} f\, d(\mu_1\otimes\mu_2) = \int_{X_1}\Big(\int_{X_2} f_{x_1}\, d\mu_2\Big)\,d\mu_1$$

$$= \int_{X_2}\Big(\int_{X_1} f_{x_2}\, d\mu_1\Big)\,d\mu_2$$

Example: $X = \mathbb{N}$, $\mathcal{A} = \mathcal{P}(\mathbb{N})$, $\mu = \nu$ counting measure

$f: X \to \mathbb{R}$, $n \mapsto f_n$ (real sequence)

$$\int f\,d\mu = \sum^{\infty} f_n$$

$$\int_X f \, d\mu = \sum_{n=1}^{\infty} f_n$$

$$g : X \times X \to \mathbb{R} , \quad (n,m) \mapsto g_{n,m}$$

$$\sum_n \sum_m g_{n,m} \stackrel{?}{=} \sum_m \sum_n g_{n,m}$$ ok, if $\sum_m \sum_n |g_{n,m}| < \infty$

$g_{n,m}$	$n=1$	2	3	4	5	
$m=1$	1	-1	0	0	0	. .
2	0	1	-1	0	0	. .
3	0	0	1	-1	0	. .
4	0	0	0	1	-1	. .
	.	.	.	.	.	

$$\sum_n g_{n,m} = 0 \quad \forall m$$

$$\sum_m \sum_n g_{n,m} = 0$$

$$\sum_m g_{n,m} = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases}$$

$$\sum_n \sum_m g_{n,m} = 1$$

Preface on the two physics lectures

A physical theory is a (mathematical) model for how (parts of) the physical world could work.

Physics is about

- inventing resp. discovering good „theories"
- collecting empirical data (experiments)
- comparing the empirical facts about our world with our theoritical worlds.

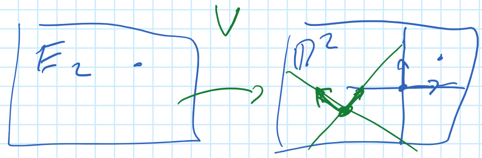
Quote from Democrit (400 BC):

„Apparently there is color, apparently sweetness, apparently bitterness, while in reality there are only atoms and empty space."

Modern physics uses mathematical models because (at least in good models) empirical predictions can be deduced by mathematical methods (computing, simulating, proving) in an unambiguous way.

Classical mechanics of point particles

Newtonian mechanics: Math. model for the motion of N particles (apples, planets, atoms, bullets, ...) in physical space and time.



Space: $\mathbb{E}^3$ three-dim. eucl. space,
$\mathbb{R}^3$ after choice of an origin and a ONB.

Time: $\mathbb{E}^1$ eucl. line, $\mathbb{R}$ after choice of origin.

Configuration space of N point particles

$$q \in \mathbb{R}^{3N} = \underbrace{\mathbb{R}^3 \times \cdots \times \mathbb{R}^3}_{N\text{-copies}} = \text{config. space}$$

$q = (q_1, q_2, \dots, q_N)$, $q_j \in \mathbb{R}^3$ position of jth particle.

In this model the „world" is completely specified by the positions of all particles at all times, i.e. by a curve $\underline{\gamma : \mathbb{R} \to \mathbb{R}^{3N},\ t \mapsto \gamma(t)}$ in configuration space.

The physical „law" is a second order ODE for γ, Newton's law:

mass matrix

$$\boxed{\underbrace{\ddot{\gamma}}_{\text{„}\gamma''\text{"}} = M^{-1} \cdot F(t, \gamma, \dot{\gamma})}$$

acceleration

force field $\sim \dot{\gamma}(t)^2$

Only the solutions to this ODE are possible worlds according to Newt. mech.

Assuming sufficient regul. of F, a unique solution is determined by specifying the positions $\gamma(t_0)$ and the velocities $\dot{\gamma}(t_0)$ at some time $t_0 \in \mathbb{R}$

$\Rightarrow$ predictions of the theory

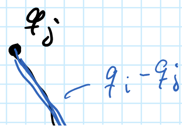
The explicit specification of M and F are also part of the law.

<u>Example</u>: <u>Gravitating bodies</u>: $M = \begin{pmatrix} m_1 & & 0 \\ & \ddots & \\ 0 & & m_N \end{pmatrix}$

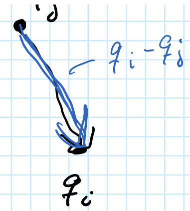
m_i = mass of the ith body,

$\|F_{ji}\| \sim \frac{1}{\|q_i - q_j\|^2}$

$$F_j(t, q, v) = F_j(q) = G \sum_i \frac{m_i m_j (q_i - q_j)}{\cdots}$$



$$F_j(t,q,v) = F_j(q) = G \sum_{i \neq j} \frac{m_i m_j \cdot (q_i - q_j)}{\|q_i - q_j\|^3}$$



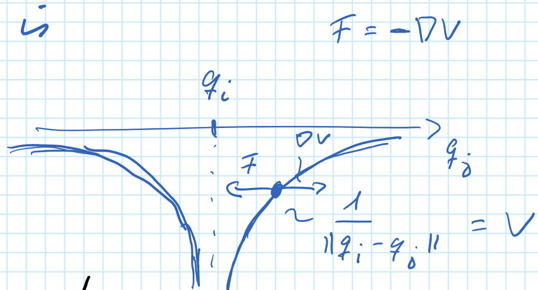
For $N=2$ one finds Kepler's ellipses as special solutions $\Rightarrow$ Kepler's laws follow from Newtonian gravitation

The gravitational force is an example of a conservative force field, i.e. a force $F : \mathbb{R}^{3N} \to \mathbb{R}^{3N}$ that is the negative gradient of a scalar fct. $V : \mathbb{R}^{3N} \to \mathbb{R}$, the so called potential, $F = -DV$.

Ex. 1: Check that the Newton pot. is

$$V(q) = -\frac{G}{2} \sum_{j \neq i} \frac{m_j m_i}{\|q_i - q_j\|}$$

double sum



For conservative forces, Newtonian mechanics displays „conservation of energy": The fcts.

$$E : \mathbb{R}^{3N} \times \mathbb{R}^{3N} \to \mathbb{R}$$

kinetic energy

$$E(q,v) := \sum_{j=1}^{N} \frac{m_j}{2} \|v_j\|^2 + V(q)$$

pot. energy

is constant along solutions of $\ddot{\gamma} = -M^{-1} DV(\gamma)$, i.e.

$$E(\gamma(t), \dot{\gamma}(t)) = E(\gamma(t_0), \dot{\gamma}(t_0)) \quad \forall t \in \mathbb{R}$$

Ex. 2: show this!

$$\frac{d}{dt}E(\gamma(t),\dot\gamma(t)) = \underbrace{\nabla_q E(\gamma(t),\dot\gamma(t))}\dot\gamma(t) + \nabla_v E(\gamma(t),\dot\gamma(t))\ddot\gamma(t)$$

$$= \langle \nabla V(\gamma(t)), \dot\gamma(t)\rangle_{\mathbb{R}^{3N}} + \sum_{j=1}^{n}\langle \underbrace{\cancel{\ } \dot\gamma_j(t)}_{\nabla_{v_j}E(\gamma,\dot\gamma)}, \overset{m_j}{\vee}\ddot\gamma_j(t)\rangle_{\mathbb{R}^3}$$

$$= \langle \dot\gamma(t), \nabla V(\gamma(t))\rangle_{\mathbb{R}^{3N}} + \langle \dot\gamma(t), \underbrace{M\ddot\gamma(t)}_{=-\nabla V(\gamma(t))}\rangle_{\mathbb{R}^{3N}}$$

Newton's equ.

$$= 0$$

$$\boxed{\text{Solutions stay on levelsets of } E \text{ !}}$$

If V is translation invariant, i.e.

$$V(q_1+a, q_2+a, \dots, q_N+a) = V(q_1,\dots,q_N) \quad \forall a\in\mathbb{R}^3$$

then also total momentum

$$P(q,v) = P(v) = \sum_{j=1}^{N} \underbrace{m_j v_j}_{=:p_j} \in \mathbb{R}^3$$

„momentum of part. j"

is conserved.

If V is invariant under rotations of $\mathbb{R}^3$,

$$V(Rq_1,\dots,Rq_N) = V(q_1,\dots,q_N)$$

then angular momentum

$$L(q,v) := \sum_{j=1}^{N} m_j\, q_j \times v_j\,.$$

is conserved. $\Rightarrow$ symmetries lead to conservation laws.

As it is the case for any 2nd order ODE, one can write Newton's equation as a first order ODE on $\mathbb{R}^{6N}$ => Hamiltonian mechanics

Another very popular and useful formalism is the Lagrangian formulation of classical mechanics as a variational problem:

A Lagrange fct. is a fct

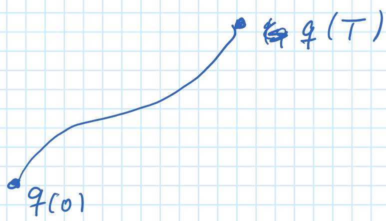
$$\mathcal{L} : \mathbb{R}^{3N} \times \mathbb{R}^{3N} \to \mathbb{R} \;,\; (q, v) \mapsto \boxed{\mathcal{L}(q, v)}$$

(e.g. $\mathcal{L}(q, v) = \sum_{j=1}^{N} \frac{m_j}{2} \| v_j \|^2 - V(q)$)

Let $\Gamma := \{ \gamma : C^2([0,T]), \mathbb{R}^{3N}) \}$ the space of C^2-paths in config-space on time interval $[0,T]$.

The action of such a path is

$$S(\gamma) := \int_0^T \mathcal{L}(\gamma(t), \dot{\gamma}(t)) \, dt$$



$$S : \Gamma \to \mathbb{R}$$

Then the principle of least action asserts that the physically possible paths γ are those for which S (when adding approp. constr.) is critical, i.e.

Euler-Lagrange

$$\boxed{D(S-\lambda\cdot H)\big|_{\gamma}=0} \qquad (*) \quad \text{Euler-Lagrange equation}$$

As
$$DS|_{\gamma}\cdot h = D_v\mathcal{L}\big|_{(\gamma(T),\dot\gamma(T))}\cdot h(T) - D_v\mathcal{L}\big|_{(\gamma(0),\dot\gamma(0))}h(0)$$
$$+\int_0^T\Big\{ D_q\mathcal{L}\big|_{(\gamma(t),\dot\gamma(t))} - \Big(\frac{d}{dt}D_v\mathcal{L}\big|_{(\gamma(t),\dot\gamma(t))}\Big)\Big\}h(t)\,dt$$

a part of $(*)$ is often (when h is only contrained at single points)

$$\boxed{D_q\mathcal{L}\big|_{(\gamma(t),\dot\gamma(t))} - \frac{d}{dt}D_v\mathcal{L}\big|_{(\gamma(t),\dot\gamma(t))} = 0 \quad \forall t}$$

For $\mathcal{L}=\sum \frac{m_j}{2}\|v_j\|^2 - V(q)$ these are exactly Newton's equations.

Hamiltonian mechanics

Phase space of N particles in $\mathbb{R}^3$ is

$$P := \mathbb{R}^{6N} = \mathbb{R}^{3N}\times\mathbb{R}^{3N},\quad x\in P \text{ is of the form}$$

$$x = (\underbrace{q_1,\dots,q_N}_{\text{position}},\underbrace{p_1,\dots,p_N}_{\text{momenta}})$$

(in general P is a symplectic space or manifold)

The canonical symplectic form on $P=\mathbb{R}^{6N}$ is

$$\varphi:\mathbb{R}^{6N}\times\mathbb{R}^{6N}\to\mathbb{R},\quad (x,x')\mapsto\langle x, Jx'\rangle$$

$$g : \mathbb{R}^{6N} \times \mathbb{R}^{6N} \to \mathbb{R}, \quad (x_1, x_2) \mapsto \langle x_1, I x_2 \rangle_{\mathbb{R}^{6N}}$$

with $$I = \begin{pmatrix} 0 & \mathrm{id}_{\mathbb{R}^{3N}} \\ -\mathrm{id}_{\mathbb{R}^{3N}} & 0 \end{pmatrix}, \quad I^T = -I$$

The law of motion is a first order ODE on P where the vector field is the symplectic gradient of a fct. $\boxed{H : P \to \mathbb{R}}$, the Hamiltonian:

$$\boxed{\dot{\alpha} = I \nabla H(\alpha)}$$

now first order ODE on phase space

$\alpha : \mathbb{R} \to P = \mathbb{R}^{6N}$

With $\alpha(t) := (\underset{\substack{\| \\ \gamma(t)}}{Q(t)}, P(t))$ this reads

$$\begin{pmatrix} \dot{Q}(t) \\ \dot{P}(t) \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{id} \\ -\mathrm{id} & 0 \end{pmatrix} \begin{pmatrix} \nabla_q H(Q(t), P(t)) \\ \nabla_p H(Q(t), P(t)) \end{pmatrix}$$

$$= \begin{pmatrix} \nabla_p H(Q(t), P(t)) \\ -\nabla_q H(Q(t), P(t)) \end{pmatrix}$$

For $$H(q,p) = \sum_{j=1}^{N} \frac{1}{2m_j} \|p_j\|^2 + V(q) \quad = E(q, M^{-1} p)$$

one finds again Newton's equat.

Let $\Phi : \mathbb{R} \times P \to P, \quad (t, x) \mapsto \alpha_x(t)$

be the flow of a Hamiltonian system.

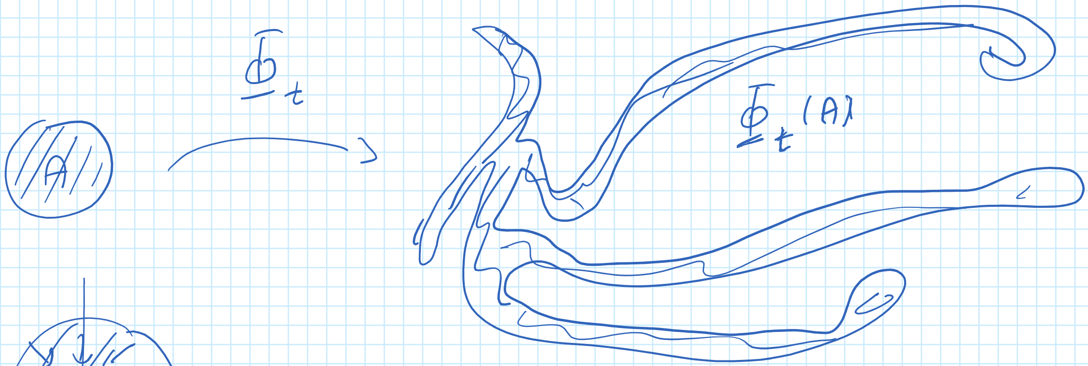
Then one has

„conservation of energy": $H \circ \Phi_t = H \quad \forall t \in \mathbb{R}$.

„conservation of phase space volume":

$\boxed{\Phi_t^* \lambda = \lambda}$ (i.e. $\lambda(\Phi_t(A)) = \lambda(A) \quad \forall A \in \mathcal{B}(P)$)

Liouville's thm (λ is Lebesgue-measure resp. Liouville measure)



$\dot{r} = -r$