8.19 2. HS Diff. u. Int.

Sei $f : [a,b] \to \mathbb{C}$ st.,

$$F(x) := \int_a^x f(t)\,dt \implies$$

$$F'(x) = f(x) \quad \forall x \in [a,b]$$

8.20 Def $F : [a,b] \to \mathbb{C}$ heißt

<u>Stammfunktion</u> von $f : [a,b] \to \mathbb{C} \iff$

F ist diffbar und $F' = f$.

8.21 Bem • 2. HS: $\forall f$ st. $\exists$ Stammfkt. F, nämlich $F(x) = \int_a^x f(t)\,dt$.

- $F(x) + C$ ist auch Stammfkt $\forall C \in \mathbb{C}$
- Das sind bereits alle Stammfktnen:

$$(F - \tilde{F})' = F' - \tilde{F}' = f - f = 0$$

$$\Rightarrow F - \tilde{F} = \text{const.}$$

8.22 Erster HS

Sei $F : [a,b] \to \mathbb{C}$ st. diffbar, $F' = f$. Dann

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Bew $\tilde{F}(x) := \int_a^x f(t)\,dt$, $\tilde{F}(a) = 0$.

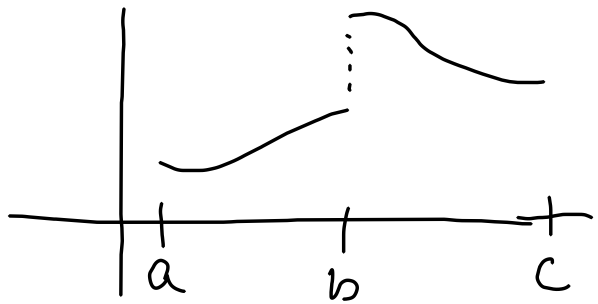
$$\tilde{F}(b) = \int_a^b f(t)\,dt$$

2. HS. $\Rightarrow$ $\tilde{F}$ ist Stammfkt

$\Longrightarrow$ $F(x) = \tilde{F}(x) + C$ $\rightarrow$ $F(b) - F(a)$

$= \tilde{F}(b) - \tilde{F}(a) = \int_a^b f(x)\,dx$. $\square$

Voraussetzung: f st.



$F_1(b) - F_1(a) + F_2(c) - F_2(b)$

Bsp $F: \mathbb{R} \to \mathbb{R}$ diffbar,

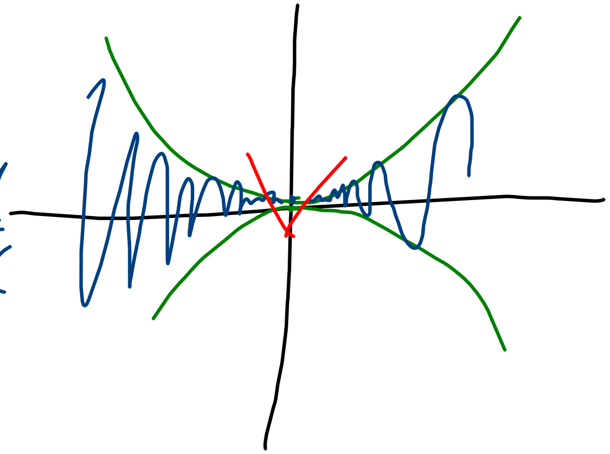
F' nicht stetig:

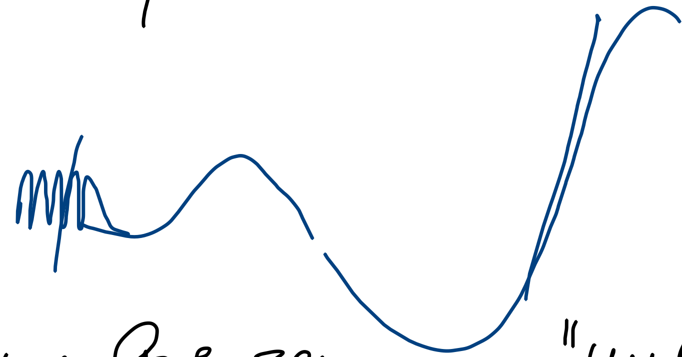
$$F(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{für } x \neq 0 \\ 0 & \end{cases}$$

$F'(0) = 0$

$x \neq 0$

$F'(x) = 2x \sin\frac{1}{x} - \cos\frac{1}{x}$





Notation $\int f(x)\,dx$ ohne Grenzen "unbestimmtes Int."

$= \text{Stammfkt } "F(x) + C"$

$= \int^x f(t)\,dt$

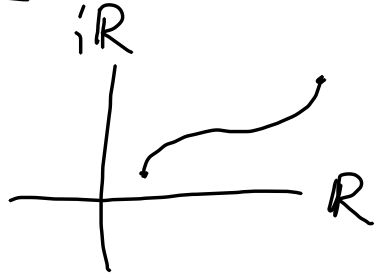
$= \int_a^x f(t)\,dt$ mit bel. a

$\int_a^b f(t)\,dt$ = "bestimmtes Int."; aber oft $\int f(x)\,dx = \int_{-\infty}^{+\infty} f(x)\,dx$

8.24 Bspe

• $0 < a < b$, $\alpha \in \mathbb{R} \setminus \{-1\}$

$$\int_a^b x^\alpha \, dx = \frac{x^{\alpha+1}}{\alpha+1}\Big|_a^b = \frac{b^{\alpha+1} - a^{\alpha+1}}{\alpha+1}$$



• $0 < a < b$

$$\int_a^b x^{-1} \, dx = \int_a^b \frac{dx}{x} = \int_a^b dx \, \frac{1}{x} = \ln x \Big|_a^b = \ln \frac{b}{a}.$$

8.25 Korollar Integralrestglied der Taylorformel

Sei $f \in C^{n+1}([a,b])$, $x, x_0 \in [a,b]$. Dann

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + \underbrace{\int_{x_0}^{x} \frac{f^{(n+1)}(t)}{n!}(x-t)^n \, dt}_{= R_n(x,x_0)}$$

(vgl. "Lagrange-Restglied"
$$R_n(x,x_0) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1} \Big)$$

<u>Beweis</u> Für $t \in [a,b]$ setze

$$F(t) := f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(t)}{k!}(x-t)^k$$

$$R_n(x,x_0) = F(x_0), \quad F \in C^1([a,b]),$$

$$F(x) = 0,$$

1. HS $\Rightarrow$ $$F(x) - F(x_0) = \int_{x_0}^{x} F'(t)\,dt$$

$$F'(t) = -\sum_{k=0}^{n} \frac{f^{(k+1)}(t)}{k!}(x-t)^k + \sum_{k=\cancel{0}\,1}^{n} \frac{f^{(k)}(t)}{\cancel{k!}\,(k-1)!}\cancel{k}(x-t)^{k-1}$$

$$= -\frac{f^{(n+1)}(t)}{n!}(x-t)^n \Longrightarrow \text{Beh.}$$

$\square$

8.2 Integrationsmethoden

8.26 Satz über partielle Integration

Seien $f, g : [a,b] \to \mathbb{C}$, f st., g st. diffbar, $F' = f$. Dann

$$\int_a^b fg = \big[Fg\big]_a^b - \int_a^b Fg'$$

Bew $\big[Fg\big]_a^b \overset{1.\,HS}{=} \int_a^b (Fg)' = \int_a^b (fg + Fg') = \int_a^b fg + \int_a^b Fg'$. $\square$

<u>Merke</u> $\int u'v = -\int uv' + [uv]$

<u>8.28 Bspe</u> a) untypisch

b) $$\int_a^b x e^x \, dx = \Big[x e^x \Big]_a^b - \underbrace{\int_a^b 1 e^x \, dx}_{[e^x]_a^b}$$

$$= \Big[x e^x - e^x \Big]_a^b .$$

$$\int_a^b x^2 e^x \, dx = \Big[x^2 e^x \Big]_a^b - \int_a^b 2x e^x \, dx$$

$$= \Big[x^2 e^x - 2x e^x + 2 e^x \Big]_a^b$$

Poly p vom Grad n: $\int_a^b p(x) e^x \, dx$ durch n-fache part. Int.

uv $\quad$ $v \quad u$

Daumenregel "DETAIL"

D = Differential wie $u'(\cancel{x})$

E = Exponential wie e^{2x}

T = trigonometrisch wie $\cos(5x)$

A = algebraisch wie x^α

I = invers wie $\arcsin x$

L = logarithmisch wie $\ln(1+x)$

c) $$\int_0^x \cos^2 y \, dy = \int_0^x \cos y \cos y \, dy$$

$$= \Big[\sin y \cos y\Big]_0^x + \int_0^x \underbrace{\sin y \sin y}_{1-\cos^2 y} \, dy$$

$$= \sin x \cos x + x - \int_0^x \cos^2 y \, dy .$$

8.29 Substitutionsregel

Seien $f:[a,b] \to \mathbb{C}$ st., $g:[c,d] \to [a,b]$ st. diffbar.

Dann $$\int_{g(c)}^{g(d)} f(y)\,dy = \int_c^d f(g(x))\,g'(x)\,dx.$$

Bew Sei $F' = f$. Dann $LS = \int_{g(c)}^{g(d)} f(y)\,dy = F(g(d)) - F(g(c))$

$$\overset{1.\,HS}{=} \int_c^d \Big(\underbrace{F(g(x))}_{F \circ g}\Big)'\,dx \underset{\text{Kettenregel}}{=} \int_c^d \big(F'(g(x))\,g'(x)\big)\,dx$$

$\square$

Merke wenn $y = g(x)$, dann "$dy = \frac{dy}{dx}\,dx = g'(x)\,dx$"

Etwa $d \sin x = \cos x\,dx$. ~~Beachte:~~

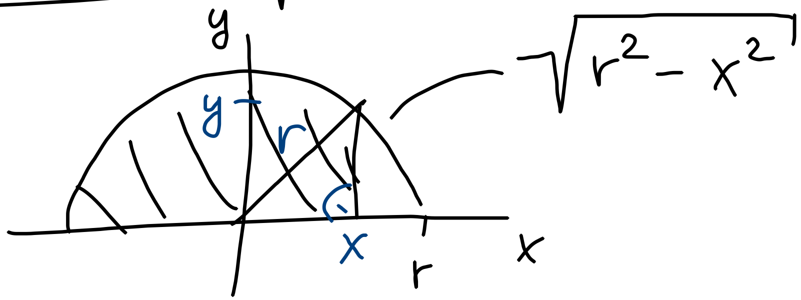
Beachte: Anwendung "vorwärts" oder "rückwärts"

8.31 Bsp Subst. rückwärts

$$\int_a^b \frac{x^3}{(x^4+1)^2}\,dx \underset{y=g(x)=x^4}{=} \int_a^b \frac{1}{(x^4+1)^2}\,\frac{dx^4}{4} = \int_{a^4}^{b^4} \frac{1}{(y+1)^2}\,\frac{dy}{4}$$

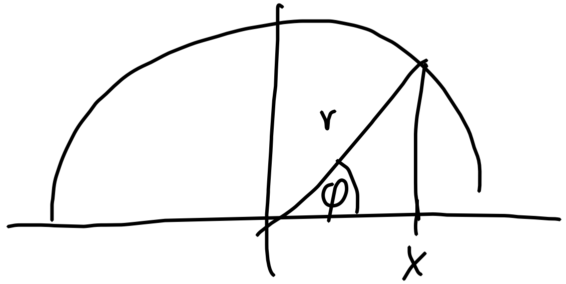
$$= \frac{1}{4}\left[-\frac{1}{y+1}\right]_{a^4}^{b^4} = \frac{1}{4(a^4+1)} - \frac{1}{4(b^4+1)}$$

8.32 Bsp Subst vorwärts: Fläche $\frac{F}{2}$ eines Halbkreises



$$\frac{F}{2} = \int_{-r}^{r} \sqrt{r^2-x^2}\,dx$$

substituiere $x = r\cos\varphi$

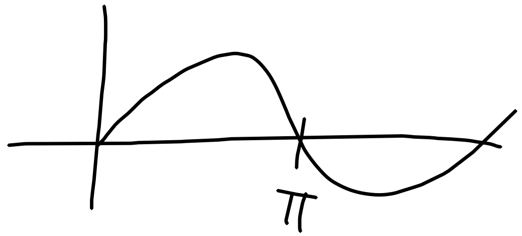


$\varphi \in [0, \pi]$ während

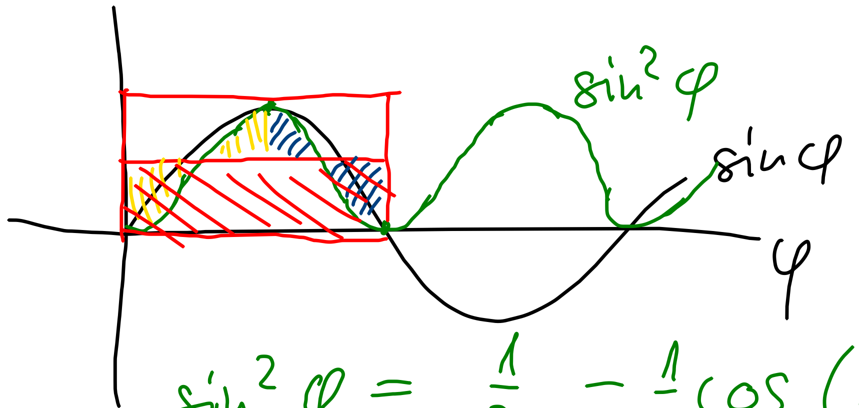
$x \in [-r, r]$, $g(\varphi) = r\cos\varphi$

$$\frac{F}{2} = \int_{-r}^{r} \sqrt{r^2 - x^2}\, dx = \int_{\pi}^{0} \sqrt{\underbrace{r^2 - r^2\cos^2\varphi}_{r^2(\underbrace{1-\cos^2\varphi}_{\sin^2\varphi})}} \; \underbrace{d(r\cos\varphi)}_{-r\sin\varphi\, d\varphi}$$

$$= -r \int_{\pi}^{0} r\sin\varphi \quad \sin\varphi\, d\varphi$$



$$= r^2 \underbrace{\int_{0}^{\pi} \sin^2\varphi\, d\varphi}_{\frac{\pi}{2}} = \frac{r^2\pi}{2}$$



$\sin^2\varphi = \frac{1}{2} - \frac{1}{2}\cos(2\varphi)$