Partialbruchzerlegung

$$\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$$

$$\frac{q(x)}{p(x)} = \frac{A}{(x+c)^n} + \ldots$$

in $\mathbb{C}$ 8.43 Satz $\operatorname{grad} p = n$, $p(z) = a_n \prod_{j=1}^{k} (z - z_j)^{l_j}$

$a_n \in \mathbb{C}\setminus\{0\}$ $\qquad$ z_j: NST in $\mathbb{C}$

$\exists A_{ij} \in \mathbb{C}$ eind. best., Poly $h(z)$:

$$\forall z \in \mathbb{C}\setminus\{z_1,\ldots,z_k\}: \quad \frac{q(z)}{p(z)} = h(z) + \sum_{j=1}^{k} \sum_{m=1}^{l_j} \frac{A_{jm}}{(z-z_j)^m}$$

$$= h(z) + \frac{A_{11}}{z-z_1} + \frac{A_{12}}{(z-z_1)^2} + \dots$$

$$\dots + \frac{A_{1\ell_1}}{(z-z_1)^{\ell_1}}$$

$$+ \frac{A_{21}}{z-z_2} + \dots$$

$$\dots + \frac{A_{k\ell_k}}{(z-z_k)^{\ell_k}}.$$

Dabei ist ~~k~~ $\operatorname{grad} h = \operatorname{grad} q - \operatorname{grad} p$

falls $\operatorname{grad} q \geq \operatorname{grad} p$

sonst ~~q~~ $h = 0$.

Bsp
$$\frac{x^4+2x}{(x+1)(x-1)^2} = \underbrace{c_1 x + c_0}_{h} + \frac{A_{11}}{x+1} + \frac{A_{21}}{x-1} + \frac{A_{22}}{(x-1)^2}.$$

<u>Bew</u> Poly div. $q = h \cdot p + r$

$$\Rightarrow \frac{q(z)}{p(z)} = h(z) + \frac{r(z)}{p(z)}$$

mit Grad $r <$ grad p oder $r = 0$.

<u>Existenz der Ajm</u>: Induktion nach $n = \text{Grad}\, p$.

<u>$n = 1$</u>: $\dfrac{r(z)}{p(z)} = \dfrac{A}{z - z_1}$

<u>$n-1 \Rightarrow n$</u>: Sei z_0 NST von p der Ordnung $\ell \geq 1$:

$$p(z) = (z - z_0)^\ell \tilde{p}(z) \quad \text{mit } \tilde{p}(z_0) \neq 0 \quad \text{Grad } \tilde{p} = n - \ell \leq n - 1$$

$\forall z \cancel{\in \mathbb{C}}$ mit $p(z) \neq 0$

Beh $\quad \dfrac{r(z)}{p(z)} = \dfrac{r(z)}{(z-z_0)^l \tilde{p}(z)} = \dfrac{A}{(z-z_0)^l} + \underbrace{\dfrac{\tilde{r}(z)}{\tilde{p}(z)(z-z_0)^{l-1}}}_{\text{nach Ind.vor. in Partialbrüche zerlegbar}}$

mit $A \in \mathbb{C}$, $\tilde{r}$ Poly mit grad < grad r

oder $\quad \dfrac{r(z)}{\tilde{p}(z)} = A + \dfrac{\tilde{r}(z)}{\tilde{p}(z)}(z-z_0) \qquad \forall z$ mit $\tilde{p}(z) \neq 0$

Bew der Beh: Setze $A = \dfrac{r(z_0)}{\tilde{p}(z_0)}$. Wegen

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \ldots + ab^{n-2} + b^{n-1})$$

gilt $\dfrac{r(z)}{\tilde{p}(z)} - A = \dfrac{1}{\tilde{p}(z)\tilde{p}(z_0)}\Big(\underbrace{r(z)\tilde{p}(z_0) - r(z_0)\tilde{p}(z)}_{\text{teilbar durch } z-z_0}\Big) = \dfrac{1}{\tilde{p}(z)}\tilde{r}(z)(z-z_0)$

$$z^n z_0^m - z_0^n z^m \underset{\text{OBdA } n>m,\ n=m+k+1}{=} (z z_0)^m (z^{k+1} - z_0^{k+1}) = (z z_0)^m (z-z_0)(\ldots) \quad \square$$

Eindeutigkeit der A_{jm}:

Zu zeigen: Wenn $\displaystyle\sum_{j=1}^{k}\sum_{m=1}^{l_j} \frac{A_{jm}}{(z-z_j)^m} = 0 \quad \forall z \in \mathbb{C}\setminus\{z_1 \dots z_k\}$,

dann alle $A_{jm} = 0$.

Bew: $$0 = \prod_{i=1}^{k}(z-z_i)^{l_i} \sum_{j=1}^{k}\sum_{m=1}^{l_j} \frac{A_{jm}}{(z-z_j)^m}$$

$$= \sum_{j=1}^{k\,\cancel{m}} \sum_{m=1}^{l_j} A_{jm}\,(z-z_j)^{l_j-m} \prod_{i\neq j}(z-z_i)^{l_i}$$

$\forall z \in \mathbb{C}$ wg. Stetigkeit. Setze $z = z_J \Longrightarrow 0 = A_{J l_J}$

$\longrightarrow$ RS teilbar durch $z - z_J \overset{St.}{\Longrightarrow}$ Quotient $\frac{RS}{z-z_J} = 0 \quad \forall z \in \mathbb{C}$

Wiederhole mit $l_J - 1$ statt l_J. $\square$

Bsp $$\frac{x^4+2x}{(x+1)(x-1)^2} = ?$$

$$\text{Nenner} = (x+1)(x-1)^2 = (x+1)(x^2-2x+1)$$
$$= x^3 + x^2 - 2x^2 - 2x + x + 1$$
$$= x^3 - x^2 - x + 1$$

Poly. div.
$$(x^4+2x) : (x^3-x^2-x+1) = x + 1 + \frac{2x^2+2x-1}{(x+1)(x-1)^2}$$
$$\underline{x^4 - x^3 - x^2 + x}$$
$$+x^3 + x^2 + \cancel{3x} + x$$
$$\underline{x^3 - x^2 - x + 1}$$
$$2x^2 + 2x - 1$$

$$\frac{2x^2+2x-1}{(x+1)(x-1)^2} = \frac{A_{11}}{x+1} + \frac{A_{21}}{x-1} + \frac{A_{22}}{(x-1)^2}$$

$$\forall x \in \mathbb{C} \setminus \{-1, 1\}$$

$\Longleftrightarrow$

$$2x^2+2x-1 = A_{11}(x-1)^2 + A_{21}(x+1)(x-1) + A_{22}(x+1)$$

$$\forall x \in \mathbb{C} \cancel{\setminus \{-1, 1\}}$$

Methode 1: Koeff.vergleich

$$RS = (A_{11} + A_{21})x^2 + (-2A_{11} - A_{21} + A_{22})x + (A_{11} - 2A_{21} + A_{22})$$

$$\overset{K.V.}{\Longrightarrow} \left\{ \begin{aligned} 2 &= A_{11} + A_{21} \\ 2 &= -2A_{11} - A_{21} + A_{22} \\ -1 &= A_{11} - 2A_{21} + A_{22} \end{aligned} \right.$$

Methode 2

$$2x^2+2x-1 = A_{11}(x-1)^2 + A_{21}(x+1)(x-1) + A_{22}(x+1)$$

setze $x = NST = \pm 1$;

oder $x =$ bel.

$$2+2-1 = 2A_{22} = 3. \quad (x=1)$$

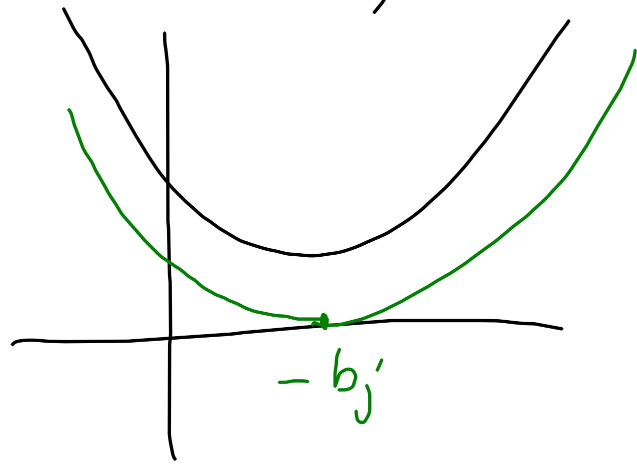
$$2-2-1 = 4A_{11} = -1 \quad (x=-1)$$

$$-1 = \underbrace{A_{11}}_{-\frac{1}{4}} - \underbrace{A_{21}}_{} + \underbrace{A_{22}}_{\frac{3}{2}} \quad (x=0)$$

PBZ in $\mathbb{R}$

8.44 Satz q, p reelle Poly

$$p(x) = a_n \prod_{j=1}^{k} (x - x_j)^{\ell_j} \prod_{j=1}^{r} (x^2 + 2b_j x + c_j)^{m_j}$$



mit $b_j^2 < c_j$ (irreduzibel, keine NST in $\mathbb{R}$)

$$\boxed{(x + b_j)^2 = x^2 + 2b_j x + b_j^2}$$

$\exists\, A_{jm}, B_{jm}, C_{jm} \in \mathbb{R}$ und best. reelles Poly $h(x)$:

$$\frac{q(x)}{p(x)} = h(x) + \sum_{j=1}^{k} \sum_{m=1}^{\ell_j} \frac{A_{jm}}{(x - x_j)^m} + \sum_{j=1}^{r} \sum_{m=1}^{m_j} \frac{B_{jm} x + C_{jm}}{(x^2 + 2b_j x + c_j)^m}.$$

Bsp

$$\frac{1}{x(x^2+1)} = \left[\frac{A_1}{x} + \frac{A_2}{x-i} + \frac{A_3}{x+i}\right]$$

$$= \frac{A}{x} + \frac{Bx+C}{x^2+1}, \quad A, B, C \in \mathbb{R}.$$

Anwendung $$\int \frac{1}{x+c}\,dx = \ln(x+c) + C$$

$m>1$: $$\int \frac{1}{(x+c)^m}\,dx = \frac{1}{-m+1}(x+c)^{-m+1} + C$$

$$\int \frac{dx}{x^2+2bx+c} \underset{y=x+b}{=} \int \frac{dy}{(y)^2 + \underbrace{(c-b^2)}_{=:\alpha^2>0}} \underset{\alpha y=u}{=}$$

$$= \int \frac{\alpha\, du}{\alpha^2 u^2 + \alpha^2} = \frac{1}{\alpha} \int \frac{du}{u^2 + 1}$$

$$= \frac{1}{\alpha} \arctan u + C$$

$$= \frac{1}{\sqrt{c - b^2}} \arctan\left(\frac{x + b}{\sqrt{c - b^2}} \right) + C$$

$$\int \frac{x\, dx}{x^2 + 1} \underset{y = x^2}{=} \frac{1}{2} \int \frac{dy}{y + 1} = \frac{1}{2} \ln |y + 1| + C$$

$$= \frac{1}{2} \ln (x^2 + 1) + C$$

$$\int \frac{x\, dx}{(x^2 + 1)^m} \overset{m > 1}{=} - \frac{1}{2(m - 1)} \frac{1}{(x^2 + 1)^{m-1}} + C$$

$$\int \frac{dx}{(x^2+1)^n} = \int \left(\frac{x^2+1}{(x^2+1)^n} - \underbrace{\frac{x^2}{(x^2+1)^n}}_{} \right) dx$$

$$x \cdot \frac{x}{(x^2+1)^n}$$

$$u \; v'$$

$$= \underbrace{\int \frac{dx}{(x^2+1)^{n-1}}}_{\text{bek.}} + \frac{1}{2(n-1)} \, \frac{x \cdot 1}{(x^2+1)^{n-1}} - \underbrace{\int \frac{1}{2(n-1)(x^2+1)^{n-1}} \, dx}_{\text{bek.}}$$