SYLLABUS FOR MATHEMATICAL QUANTUM THEORY

WINTER TERM 2021/2022

This course is offered by the Institute of Mathematics as a core course in the Master Program in Mathematical Physics. Students enrolled in the regular mathematics or physics programs at Tübingen (or similarly prepared students) are also welcome.

Instructor: Prof. Dr. Marius Lemm
Office: C Building, 4P39
Email: `marius.lemm[at]uni-tuebingen.ch`

Teaching Assistant: Dr. Oliver Siebert
Office: C Building, 4P42
Email: `oliver.siebert[at]uni-tuebingen.ch`

Course logistics

The ILIAS website is the central hub for all course matters. You can access it with the course password "mqt2021".

Lectures.
Mondays 10:15 -1:45 in room H2C14 **(note the change!)** and Thursdays 12:15 - 13:45 in room C4H33. Lectures will be delivered in a *hybrid format*: students will be able to attend either on-site or online via Zoom. Per university guidelines, on-site attendance requires 3G health status certification.

All lectures will be reachable under this MQT Zoom link.
The meeting ID is 947 2271 2656 and the passcode is 717274

Exercise session.
Tuesdays 14:15 - 15:45 in room A6 G07. This is an excellent opportunity to directly ask the teaching assistant Oliver Siebert questions about the course material and the homework problems.

Discord server.
The course will use a Discord server to enable discussion and exchange among students, local and remote. Forming study groups is strongly encouraged. Here is the sign-up link for the MQT discord server.

Date: October 19, 2021.

Homework.
Students will complete a weekly problem set which is due every Thursday at 12:00 noon. Each week, the homework problems will be graded by the teaching assistant.

Admittance to the final exam requires receiving at least 50% of the overall homework points. The lowest homework score will be dropped automatically.

Students are strongly encouraged to collaborate on the problem sets, which can be organized for example via discord. Every student should write up and submit their own solution to get accurate feedback. You will submit your homework through the URM student portal, so please register for the exercise session of the MQT class there.

Final exam.
At the end of the term, there will be a written final exam. Further details to be announced later. Please use the university's ALMA website to register for the final exam.

Course material

Course goals.
This course presents the mathematical formalism that underlies quantum mechanics. Quantum mechanics describes most of the matter around us and is one of the most successful physical theories. This course teaches the mathematical framework from functional analysis and spectral theory that grounds quantum mechanics rigorously. A central role is played by the theory of unbounded operators on Hilbert spaces, especially the spectral theorem for unbounded, self-adjoint operators. Along the way, the derived results are used to make various physical concepts mathematically precise, e.g., wave functions, observables, Schrödinger operators, continuous vs. discrete spectrum, and quantum dynamics.

The course serves both as an advanced course in operator theory and as an introduction to mathematical physics.

Prerequisites.
A firm background in mathematical analysis is required. This includes working knowledge of measure theory (especially L^p spaces) and basic functional analysis (especially Hilbert spaces). Familiarity with tools from partial differential equations, in particular Sobolev spaces and the Fourier transform, is also useful.

Having previously taken a physics class on quantum mechanics is very helpful for understanding the motivations behind the results, but not strictly necessary.

Outline of the course.
In the first week, we will review L^p spaces and Sobolev spaces. We will also introduce the basic ingredients of quantum mechanics: The wave function and the Schrödinger equation. The course is comprised of the following three main units.

The *first main unit* takes a closer look at the free Schrödinger equation. To understand its solutions, we develop the general theory of the Fourier transform. We briefly contrast solutions to the free Schrödinger equation with solutions to the heat equation. An application is that we can define the free Schrödinger evolution for arbitrary L^2 initial data.

The *second unit* concerns the general theory of unbounded operators. This part begins with a quick review of basic Hilbert space theory including the definition of adjoint operator in Hilbert space. The highlight of this unit is the proof of the spectral theorem for unbounded, self-adjoint operators and the associated functional calculus. With the spectral theorem at hand, we can rigorously state a general version of the postulates of quantum mechanics. We then show how different spectral types lead to different quantum dynamics ("RAGE theorem").

The *third unit* focuses on Schrödinger operators $-\Delta + V(x)$. For physical reasons, it is important that these operators are realized in a self-adjoint fashion and so we will start by discussing criteria for self-adjointness. Afterwards, we prove Weyl's theorem that relatively compact perturbations cannot change the essential spectrum. Finally, depending on time, we study the associated energy functional and prove some basic results for this important variational problem, such as existence and uniqueness of ground states and the min-max characterization of eigenvalues.

Resources.
The course will loosely follow a set of 2018/2019 lecture notes by Marcello Porta for a similar class. These are available from day 1 on the course's ILIAS website.

In addition to the lecture notes, the following textbooks are recommended for (completely optional) further reading.

(1) M. Reed and B. Simon, *Methods of Modern Mathematical Physics: I Functional Analysis*, second edition, Academic Press, 1980
(2) M. Reed and B. Simon, *Methods of Modern Mathematical Physics: II Fourier Analysis, Self Adjointness*, Academic Press, 1975

— *Short description* — These two books are part of a classic four-volume series by Reed and Simon. They have a unique style that effectively blends intuition with key details. Overall, they contain significantly more mathematical theory than we will cover (e.g. we will not explore general locally convex spaces).

(3) E.H. Lieb and M. Loss, *Analysis*, Graduate Studies in Mathematics **14**, AMS, 2001

— *Short description* — This book by Lieb and Loss develops some of the most useful tools of analysis with very little fuss or abstraction by focusing completely on concrete function spaces over $\mathbb{R}^d$ and $\mathbb{C}^d$. The authors are mathematical physicists and a number of applications to quantum theory are included.

(4) G. Teschl, *Mathematical Methods in Quantum Mechanics: With Applications to Schrödinger Operators*, Graduate Studies in Mathematics **99**, AMS, 2009

— *Short description* — This more recent book by Teschl focuses especially on spectral theory and Schrödinger operators and therefore has substantial overlap with the material covered in this course.