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Foundations of Quantum Mechanics

In-class problems for the exercise class

Problem 5: Conserved quantities
Recall the equation of motion of Newtonian mechanics:

$$m_i \frac{d^2 \boldsymbol{Q}_i}{dt^2} = -\nabla_i V(\boldsymbol{Q}_1, \ldots, \boldsymbol{Q}_N)\,. \tag{1}$$

Suppose that V is invariant under rotations and translations:

$$V\Big(R\boldsymbol{Q}_1 + \boldsymbol{a}, \ldots, R\boldsymbol{Q}_N + \boldsymbol{a}\Big) = V\big(\boldsymbol{Q}_1, \ldots, \boldsymbol{Q}_N\big) \tag{2}$$

for all $\boldsymbol{a} \in \mathbb{R}^3$ and $R \in SO(3)$. Show that

$$\text{the energy } E = \sum_{k=1}^{N} \frac{m_k}{2} \boldsymbol{v}_k^2 + V(\boldsymbol{Q}_1, \ldots, \boldsymbol{Q}_N) \tag{3}$$

$$\text{the momentum } \boldsymbol{p} = \sum_{k=1}^{N} m_k \boldsymbol{v}_k \tag{4}$$

$$\text{the angular momentum } \boldsymbol{L} = \sum_{k=1}^{N} m_k \boldsymbol{Q}_k \times \boldsymbol{v}_k \tag{5}$$

are conserved.

Problem 6: Galilean relativity
A Galilean change of space-time coordinates ("Galilean boost") is given by

$$\boldsymbol{x}' = \boldsymbol{x} + \boldsymbol{v}t\,, \quad t' = t \tag{6}$$

with a constant $\boldsymbol{v} \in \mathbb{R}^3$ called the relative velocity.

(a) Show that if V is translation invariant then Newton's equation of motion is invariant under Galilean boosts: If $t \mapsto (\boldsymbol{Q}_1, \ldots, \boldsymbol{Q}_N)$ is a solution then so is $t \mapsto (\boldsymbol{Q}'_1, \ldots, \boldsymbol{Q}'_N)$.

(b) Show that if V is translation invariant and $\psi(t, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_N)$ is a solution of the Schrödinger equation, then so is

$$\psi'(t', \boldsymbol{x}'_1, \ldots, \boldsymbol{x}'_N) = \exp\left[\tfrac{i}{\hbar} \sum_{i=1}^{N} m_i (\boldsymbol{x}'_i \cdot \boldsymbol{v} - \tfrac{1}{2}\boldsymbol{v}^2 t')\right] \psi\Big(t', \boldsymbol{x}'_1 - \boldsymbol{v}t', \ldots, \boldsymbol{x}'_N - \boldsymbol{v}t'\Big). \tag{7}$$

Problem 7: Polarization identity
Verify that

$$\langle \psi | \phi \rangle = \tfrac{1}{4}\Big(\|\psi + \phi\|^2 - \|\psi - \phi\|^2 - i\|\psi + i\phi\|^2 + i\|\psi - i\phi\|^2\Big), \tag{8}$$

using the properties of an inner product. The polarization identity allows us to express inner products in terms of norms.