Foundations of Quantum Mechanics: Assignment 4

Exercise 14: Essay question. Describe the delayed-choice double-slit experiment. Why does it seem paradoxical? How does the paradox get resolved in Bohmian mechanics?

Exercise 15: Fourier transform
(a) Find the Fourier transform $\widehat{\psi}$ of the function

$$\psi(x) = \begin{cases} 0 & x < -1 \\ 2^{-1/2} & -1 \leq x \leq 1 \\ 0 & x > 1, \end{cases} \tag{1}$$

where x is a 1-dimensional variable.

(b) (optional) Plot $\widehat{\psi}$ using a computer.

(c) (optional) Using suitable software (such as Mathematica, Maple, or Matlab), make the computer find the formula for $\widehat{\psi}$. (That is, you need to find the command for symbolically computing Fourier transforms and the one for defining a function like (1), and run them.)

Exercise 16: Pauli matrices
The three *Pauli matrices* are

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2}$$

(a) For each of σ_1 and σ_2, find an orthonormal basis of eigenvectors in $\mathbb{C}^2$.

(b) Show that for every unit vector $\boldsymbol{n} \in \mathbb{R}^3$, the Pauli matrix in direction $\boldsymbol{n}$, $\sigma_{\boldsymbol{n}} := \boldsymbol{n} \cdot \boldsymbol{\sigma} = n_1\sigma_1 + n_2\sigma_2 + n_3\sigma_3$, has eigenvalues ± 1. (Hint: compute det and trace.)

(c) Show that every self-adjoint complex 2×2 matrix A is of the form $A = cI + \boldsymbol{u} \cdot \boldsymbol{\sigma}$ with $c \in \mathbb{R}$ and $\boldsymbol{u} \in \mathbb{R}^3$.

Exercise 17: Potential step
Consider a potential step $V(x) = V_0 \, 1_{0 \leq x}$ in 1d with $V_0 > 0$. We want to compute the reflection and transmission probabilities as a function of the incoming momentum $\hbar k_0$, assuming that $E := \hbar^2 k_0^2/2m > V_0$. A recipe for that is to consider a plane wave e^{ik_0x} coming from $x = -\infty$ and see how much gets reflected and transmitted by constructing an eigenfunction ψ of H, $H\psi = E\psi$, from the ansatz

$$\psi(x) = \begin{cases} Ae^{ik_0x} + Be^{-ik_0x} & \text{for } x < 0 \\ Ce^{iK_0x} & \text{for } x > 0 \end{cases} \tag{3}$$

with complex coefficients A, B, C, representing the incoming wave e^{ik_0x}, the reflected wave e^{-ik_0x}, and the transmitted wave e^{iK_0x}. Regarding V as a limit of continuous functions leads to the further requirement that ψ be continuous and continuously differentiable at 0.

(a) Determine K_0 from $H\psi = E\psi$.

(b) Determine B and C from A.

(c) The recipe says that the three waves are associated with probability currents $j_{\text{in}} = \hbar k_0 |A|^2/m$, $j_R = \hbar k_0 |B|^2/m$, and $j_T = \hbar K_0 |C|^2/m$; and that the reflection probability is $P_R = j_R/j_{\text{in}}$, while the transmission probability is $P_T = j_T/j_{\text{in}}$. Compute P_R and P_T.

(d) To justify the recipe, consider a "plane wave packet" $\psi(x) = \phi(x)\, e^{ik_0x}$ with a (non-Gaussian) envelope profile $\phi(x)$ that is nearly constant over a region of length L much larger than the wave length $2\pi/k_0$ and then drops to 0 over a length much smaller than L but still much larger than the wave length. Let us take for granted that under the free Schrödinger evolution the envelope function ϕ will approximately maintain its shape (in particular, its length) for a long time and simply move at speed $\hbar k_0/m$. Suppose the packet hits the step at $t = 0$; forget about what happens at the edges of the packet and focus on the bulk. A reflected plane wave packet and a transmitted one are generated; at time $\tau > 0$, the incoming plane wave packet is used up, and the two outgoing packets end. During $0 < t < \tau$, the region around 0 is well approximated by (3); after τ, the outgoing packets keep moving away from the origin. Determine τ and the lengths L_R and L_T of the reflected and transmitted packets ψ_R and ψ_T.

(e) Determine $P_R = \|\psi_R\|^2$ and $P_T = \|\psi_T\|^2$ and verify that they agree with part (c).

(f) Draw a space-time diagram of the Bohmian trajectories in the bulk (that is, ignoring any edge effects).

Hand in: by Tuesday, November 23, 2021, 8:15am via `urm.math.uni-tuebingen.de`

No reading assignment this week.