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Winter semester 2021/22
November 12, 2021

Foundations of QM: In-Class Problems

Let x be a 1-d variable and $g_\sigma(x)$ the Gaussian probability density,

$$g_\sigma(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} \,. \tag{1}$$

The *Dirac δ function* can be defined heuristically as

$$\delta(x) = \lim_{\sigma\to 0} g_\sigma(x) \,. \tag{2}$$

Since $\delta(x) = 0$ for $x \neq 0$ and $\delta(0) = \infty$, the δ function is not a function in the ordinary sense; it is called a *distribution*. Based on the heuristic (2), one defines

$$\int_{\mathbb{R}} \delta(x-a)\, f(x)\, dx := \lim_{\sigma\to 0} \int_{\mathbb{R}} g_\sigma(x-a)\, f(x)\, dx \,. \tag{3}$$

It follows that if the function f is continuous at a, then

$$\int_{\mathbb{R}} \delta(x-a)\, f(x)\, dx = f(a) \,. \tag{4}$$

Mathematicians take this as the definition of the δ distribution; that is, they define $\delta(\,\cdot\, - a)$ as a linear operator from some function space such as $\mathscr{S}$ (Schwartz space) to $\mathbb{C}$, $f \mapsto f(a)$.

Problem 8: Dirac delta function (introductory level)
(a) One defines the Fourier transform $\widehat{\delta_a}(k)$ of $\delta_a(x) = \delta(x-a)$ by applying the usual integral formula. Find $\widehat{\delta_a}(k)$ for arbitrary constant $a \in \mathbb{R}$, and find the function ψ whose Fourier transform is $\widehat{\psi}(k) = \delta(k-b)$ with arbitrary constant $b \in \mathbb{R}$.

(b) One defines the derivative δ' of the δ function heuristically by

$$\delta'(x) = \lim_{\sigma\to 0} g'_\sigma(x) \tag{5}$$

and its integrals by

$$\int_{\mathbb{R}} \delta'(x-a)\, f(x)\, dx := \lim_{\sigma\to 0} \int_{\mathbb{R}} g'_\sigma(x-a)\, f(x)\, dx \,. \tag{6}$$

Sketch g'_σ for small σ.

(c) Using integration by parts and (4), show that (for $f \in \mathscr{S}$)

$$\int_{\mathbb{R}} \delta'(x-a)\, f(x)\, dx = -f'(a) \,. \tag{7}$$

(d) Show that $\lim_{\varepsilon\to 0} \frac{1}{2\varepsilon}\big(\delta(x+\varepsilon) - \delta(x-\varepsilon)\big) = \delta'(x)$.

(e) For $\alpha \in \mathbb{R} \setminus \{0\}$, show that $\int_{\mathbb{R}} \delta(\alpha x)\, f(x)\, dx = \frac{1}{|\alpha|} f(0)$.

Problem 9: Delta function in higher dimension
(a) The d-dimensional Dirac delta function is defined by

$$\delta^d(\boldsymbol{x}-\boldsymbol{a}) = \delta(x_1-a_1)\cdots\delta(x_d-a_d) \tag{8}$$

Instead of $\delta^d(\boldsymbol{x}-\boldsymbol{a})$, one sometimes simply writes $\delta(\boldsymbol{x}-\boldsymbol{a})$. Verify that

$$\int_{\mathbb{R}^d} \delta^d(\boldsymbol{x}-\boldsymbol{a})\, f(\boldsymbol{x})\, d^d\boldsymbol{x} = f(\boldsymbol{a})\,. \tag{9}$$

(b) For a generalized orthonormal basis (GONB) with continuous parameter, $\{\phi_{\boldsymbol{k}} : \boldsymbol{k} \in \mathbb{R}^d\}$, one requires that

$$\langle \phi_{\boldsymbol{k}_1} | \phi_{\boldsymbol{k}_2} \rangle = \delta^d(\boldsymbol{k}_1 - \boldsymbol{k}_2)\,. \tag{10}$$

Verify this relation for the basis functions of Fourier transformation,

$$\phi_{\boldsymbol{k}}(\boldsymbol{x}) = (2\pi)^{-d/2} e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\,. \tag{11}$$

(c) Verify that every $\phi_{\boldsymbol{k}}$ as given by (11) is an eigenfunction of each momentum operator $P_j = -i\hbar\partial/\partial x_j$, $j = 1,\ldots,d$. Thus, (11) defines a GONB that simultaneously diagonalizes all P_j. It is therefore called the *momentum basis*.

(d) Now consider another basis, given by

$$\phi_{\boldsymbol{y}}(\boldsymbol{x}) = \delta^d(\boldsymbol{x}-\boldsymbol{y})\,. \tag{12}$$

Verify Eq. (10) for these functions. Then verify that every $\phi_{\boldsymbol{y}}$ is an eigenfunction of each position operator $X_j\psi(\boldsymbol{x}) = x_j\psi(\boldsymbol{x})$, $j = 1,\ldots,d$. Thus, (12) defines a GONB that simultaneously diagonalizes all X_j. It is therefore called the *position basis*.

Problem 10: Distributional solutions of the Schrödinger equation
While we have considered so far (and will mostly consider) only solutions ψ_t of the Schrödinger equation with initial data $\psi_0 \in L^2(\mathbb{R}^d, \mathbb{C}^m)$, it is possible to define solutions for initial data that are distributions, at least for the free Schrödinger equation: For any distribution T, take its Fourier transform, multiply by the appropriate function of $\boldsymbol{k}$ and t, and Fourier transform back. Find the Fourier transform of ψ_t if $\psi_0(\boldsymbol{x}) = \delta(\boldsymbol{x}-\boldsymbol{a})$.

Problem 11: For mathematicians
In order to make Problem 10 work rigorously, one needs (i) a suitable space $\mathscr{D}$ containing "all" distributions; (ii) a definition of the Fourier transform $\widehat{T} \in \mathscr{D}$ for every $T \in \mathscr{D}$; and (iii) that a multiplication operator M_φ with a function of the form $\varphi(\boldsymbol{k}) = \exp(i\,\text{polynomial}(\boldsymbol{k}))$ maps $\mathscr{D}$ to itself. To this end, one considers the Schwartz space $\mathscr{S}$ of rapidly decaying functions, i.e., those ψ with $|\partial^{\boldsymbol{\alpha}}\psi| < C_{n,\boldsymbol{\alpha}}|\boldsymbol{x}|^{-n}$ for all $\boldsymbol{x} \in \mathbb{R}^d, \boldsymbol{\alpha} \in \mathbb{N}_0^d, n \in \mathbb{N}_0$; one endows $\mathscr{S}$ with the topology in which $\psi_n \to \psi$ iff

$$\Big\|\psi_n - \psi\Big\|_{\boldsymbol{\alpha},\boldsymbol{\beta}} := \sup_{\boldsymbol{x}\in\mathbb{R}^d}\Big|\boldsymbol{x}^{\boldsymbol{\alpha}}\partial^{\boldsymbol{\beta}}(\psi_n - \psi)(\boldsymbol{x})\Big| \xrightarrow{n\to\infty} 0 \text{ for all } \boldsymbol{\alpha},\boldsymbol{\beta} \in \mathbb{N}_0^d,$$

where $\boldsymbol{x}^{\boldsymbol{\alpha}} = x_1^{\alpha_1}\cdots x_d^{\alpha_d}$ and $\partial^{\boldsymbol{\beta}} = \partial_1^{\beta_1}\cdots\partial_d^{\beta_d}$; one defines $\mathscr{D}$ as the *continuous dual space* $\mathscr{S}'$, i.e., the set of continuous $\mathbb{C}$-linear mappings $T : \mathscr{S} \to \mathbb{C}$; $\mathscr{S}'$ is called the *space of tempered distributions*; and one uses that the Fourier transformation $\mathscr{F}$ (as well as $\mathscr{F}^{-1}$) maps $\mathscr{S}$ to itself and is continuous as a mapping $\mathscr{S} \to \mathscr{S}$.

(a) Prove that the δ distribution is a continuous mapping from $\mathscr{S}$ to $\mathbb{C}$.

(b) Let $C^\infty_{\text{poly}}(\mathbb{R}^d)$ be the space of smooth functions φ such that φ and all of its derivatives are bounded by polynomials, $|\partial^{\boldsymbol{\alpha}}\varphi(\boldsymbol{k})| \leq |P_{\boldsymbol{\alpha}}(\boldsymbol{k})|$ for a suitable polynomial $P_{\boldsymbol{\alpha}}$. Show that for every such φ, the multiplication operator M_φ is a continuous operator $\mathscr{S} \to \mathscr{S}$.

Problem 12: Bohmian trajectories for plane waves (easy)
Let ψ_t be a plane wave solution of the Schrödinger equation with wave vector $\boldsymbol{k}$. Show that for every constant vector $\boldsymbol{a} \in \mathbb{R}^3$,

$$\boldsymbol{Q}(t) = \boldsymbol{a} + \frac{\hbar\boldsymbol{k}}{m}t \tag{13}$$

is a Bohmian trajectory with initial position $\boldsymbol{Q}(0) = \boldsymbol{a}$.

Problem 13: Galilean relativity of Bohmian mechanics
Consider again the Galilean boost

$$\boldsymbol{x}' = \boldsymbol{x} + \boldsymbol{v}t\,, \quad t' = t \tag{14}$$

with a constant $\boldsymbol{v} \in \mathbb{R}^3$. Suppose that the potential V is translation invariant and use in-class Problem 6b to show that if $t \mapsto (\boldsymbol{Q}_1, \ldots, \boldsymbol{Q}_N)$ is a solution of Bohmian mechanics then so is $t \mapsto (\boldsymbol{Q}'_1, \ldots, \boldsymbol{Q}'_N)$.