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Foundations of Quantum Mechanics: Assignment 5

Exercise 18: Essay question. Describe what the Heisenberg uncertainty relation asserts.

Exercise 19: Spectral theorem
In this problem we use the following generalization of the spectral theorem (formulated here in finite dimension): If the self-adjoint $d \times d$ matrices A and B commute, then they can be simultaneously unitarily diagonalized, i.e., there is an orthonormal basis $\{\phi_1, \ldots, \phi_d\}$ such that each ϕ_j is an eigenvector of A and an eigenvector of B.

Show that a $d \times d$ matrix C can be unitarily diagonalized iff C commutes with $C^\dagger$. Such a matrix is called "normal." (Hint: write $C = A + iB$.)

Exercise 20: Spinors
Verify that $|\boldsymbol{\omega}(\phi)| = \|\phi\|_S^2 = \phi^*\phi$. Proceed as follows: By (9.6), $\boldsymbol{\omega}(z\phi) = |z|^2\boldsymbol{\omega}(\phi)$, it suffices to show that unit spinors are associated with unit vectors. By (9.6) again, it suffices to consider ϕ with $\phi_1 \in \mathbb{R}$ (else replace ϕ by $e^{i\theta}\phi$ with appropriate θ). So we can assume, without loss of generality, $\phi = (\cos\alpha, e^{i\beta}\sin\alpha)$ with $\alpha, \beta \in \mathbb{R}$. Evaluate $\phi^*\boldsymbol{\sigma}\phi$ explicitly in terms of α and β, using the explicit formulas (9.3) for $\boldsymbol{\sigma}$. Then check that it is a unit vector.

Exercise 21: Half Angles
(a) Show that for unit vectors ϕ, χ in spin space S,

$$2\big|\langle\phi|\chi\rangle\big|^2 = 1 + \sum_{a=1}^{3} \langle\phi|\sigma_a\phi\rangle\langle\chi|\sigma_a\chi\rangle\,.$$

(b) Conclude further that if ϕ and χ have angle $\theta = \arccos\big|\langle\phi|\chi\rangle\big|$ in S, then $\boldsymbol{\omega}(\phi)$ and $\boldsymbol{\omega}(\chi)$ have angle 2θ in $\mathbb{R}^3$.

Exercise 22: Can't Distinguish Non-Orthogonal State Vectors
(a) Alice gives to Bob a single particle whose spin state ψ is either $(1,0)$ or $(0,1)$ or $\frac{1}{\sqrt{2}}(1,1)$. Bob can carry out a quantum measurement of an arbitrary self-adjoint operator. Show that it is impossible for Bob to decide with certainty which of the three states ψ is.

(b) The same with only $(1,0)$ and $\frac{1}{\sqrt{2}}(1,1)$.

Hand in: By Tuesday November 30, 2021, 8:15 am via `urm.math.uni-tuebingen.de`.

Reading assignment due Thursday December 2, 2021: T. Maudlin, Three Measurement Problems. *Topoi* **14(1)**: 7–15 (1995). Read pages 7–12 and the first two paragraphs on page 13.