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Winter semester 2021/22
December 14, 2021

Foundations of Quantum Mechanics: Assignment 8

Exercise 31: Essay question. Why does GRW theory make approximately the same predictions as the quantum formalism?

Exercise 32: Boundary conditions
On the half axis $(-\infty, 0]$, consider the Schrödinger equation $i\hbar\partial\psi/\partial t = -(\hbar^2/2m)\partial^2\psi/\partial x^2$ with boundary condition

$$\alpha \frac{\partial\psi}{\partial x}(x=0) + \beta\psi(x=0) = 0 \tag{1}$$

with constants $\alpha, \beta \in \mathbb{C}$. For $\alpha = 0, \beta = 1$ this is called a *Dirichlet boundary condition*, for $\alpha = 1$ and $\beta = 0$ a *Neumann boundary condition*. [This is Carl Neumann (1832–1925; in Tübingen 1865–1868), not John von Neumann (1903–1957).] For general $(\alpha, \beta) \neq (0, 0)$ it is called a *Robin boundary condition*. Which choices of (α, β) imply that $j(x = 0) = 0$? (They are reflecting boundary conditions and lead to a unitary time evolution.) Which imply that $j(x = 0) > 0$ whenever $\psi(x = 0) \neq 0$? (They are absorbing boundary conditions and lead to loss of probability.)

Exercise 33: Quantum Zeno effect
Zeno of Elea (c. 490–c. 430 BCE) was a Greek philosopher who claimed that motion and time cannot exist because they are inherently paradoxical notions, a claim which he tried to support by formulating various paradoxes, including one involving Achilles and a turtle. In modern times, Alan Turing (of computer science fame, lived 1912–1954) reportedly first discovered the following effect, which was later named after Zeno because of its paradoxical flavor: Suppose a quantum particle moves in 1d, and its initial wave function $\psi_0(x)$ is concentrated in the negative half axis $(-\infty, 0)$. We want to model, as a kind of time measurement, a detector, located at the origin, that clicks when the particle arrives. To this end, we imagine that the detector performs, at times $n\tau$ with $n \in \mathbb{N}$ and time resolution $\tau > 0$, a quantum measurement of $1_{x\geq 0}$, i.e., of whether the particle is in the right half axis. The ideal detector would seem to correspond to the limit $\tau \to 0$; however, in this limit, the probability that the detector *ever* clicks is 0. "A watched pot never boils," wrote Misra und Sudarshan.[1]

Prove the following simplified version: In a 2d Hilbert space $\mathbb{C}^2$, let $\psi_0 = (1, 0)$ evolve with Hamiltonian $H = \sigma_1$, interrupted by a quantum measurement of σ_3 at times $n\tau$ for all $n \in \mathbb{N}$. For any fixed $T > 0$, the probability that any of the $\approx T/\tau$ measurements in the time interval $[0, T]$ yields the result -1 tends to 0 as $\tau \to 0$.

Please turn over.

[1] B. Misra and E.C.G. Sudarshan: The Zeno's paradox in quantum theory. *Journal of Mathematical Physics* **18**: 756–763 (1977)

Exercise 34: No-cloning theorem
We show that it is impossible to duplicate the quantum state of an object without destroying the original quantum state. Let $\mathbb{S}(\mathscr{H}) = \{\psi \in \mathscr{H} : \|\psi\| = 1\}$ denote the unit sphere in $\mathscr{H}$. A *cloning mechanism* for the Hilbert space $\mathscr{H}_{\mathrm{obj}}$ would consist of a Hilbert space $\mathscr{H}_{\mathrm{app}}$, a ready state $\phi_0 \in \mathbb{S}(\mathscr{H}_{\mathrm{app}})$ of the apparatus, a ready state $\psi_0 \in \mathbb{S}(\mathscr{H}_{\mathrm{obj}})$ of the copy, and a unitary time evolution U on $\mathscr{H}_{\mathrm{obj}} \otimes \mathscr{H}_{\mathrm{obj}} \otimes \mathscr{H}_{\mathrm{app}}$ such that, for all $\psi \in \mathbb{S}(\mathscr{H}_{\mathrm{obj}})$,

$$U(\psi \otimes \psi_0 \otimes \phi_0) = \psi \otimes \psi \otimes \phi_\psi \tag{2}$$

with some $\phi_\psi \in \mathscr{H}_{\mathrm{app}}$ that may depend on ψ. Prove that if $\dim \mathscr{H}_{\mathrm{obj}} \geq 2$, then no cloning mechanism exists. (*Hint*: Consider $\psi_1 \perp \psi_2$ and $\psi_3 = \frac{1}{\sqrt{2}}\psi_1 + \frac{1}{\sqrt{2}}\psi_2$.)

Hand in: by Tuesday January 11, 2022, 8:15am via `urm.math.uni-tuebingen.de`

Reading assignment due Thursday January 13, 2022:
J. Bell: Six possible worlds of quantum mechanics. Talk given at the Symposium "Possible Worlds in Arts and Sciences" (1986), reprinted in *Speakable and Unspeakable in Quantum Mechanics* (1987), pages 181–195.