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# Foundations of Quantum Mechanics: Assignment 9

**Exercise 35: Essay question.** Explain why Schrödinger's theory Sm has a many-worlds character.

**Exercise 36: Marginal and conditional distribution**
Consider two random variables $X, Y$ that assume only values $\pm 1$. Their joint distribution can be described by a $2 \times 2$ table of probabilities. **(a)** Give a generic example of such a table (i.e., one without symmetries). For your table, compute **(b)** the marginal distribution of $X$ and **(c)** that of $Y$, as well as **(d)** the conditional distribution of $X$, given that $Y = +1$, **(e)** the expectation value $\mathbb{E}(X)$, and **(f)** $\mathbb{E}(XY)$.

**Exercise 37: Spin singlet state**
Verify through direct computation that in the spin space $\mathbb{C}^4 = \mathbb{C}^2 \otimes \mathbb{C}^2$ of two spin-$\frac{1}{2}$ particles,

$$\begin{aligned} &|x\text{-up}\rangle|x\text{-down}\rangle - |x\text{-down}\rangle|x\text{-up}\rangle \\ =\, &|y\text{-up}\rangle|y\text{-down}\rangle - |y\text{-down}\rangle|y\text{-up}\rangle \\ =\, &|z\text{-up}\rangle|z\text{-down}\rangle - |z\text{-down}\rangle|z\text{-up}\rangle \end{aligned} \tag{1}$$

up to phase factors.

**Exercise 38: Distinguish ensembles**
A source generates

**(a)** either 10,000 particle pairs in the spin singlet state
$\frac{1}{\sqrt{2}}\big(\big|z\text{-up}\big\rangle\big|z\text{-down}\big\rangle - \big|z\text{-down}\big\rangle\big|z\text{-up}\big\rangle\big) = \frac{1}{\sqrt{2}}\big(\big|x\text{-up}\big\rangle\big|x\text{-down}\big\rangle - \big|x\text{-down}\big\rangle\big|x\text{-up}\big\rangle\big)$

**(b)** or, in a random order, 5,000 pairs in $\big|z\text{-up}\big\rangle\big|z\text{-down}\big\rangle$ and 5,000 in $\big|z\text{-down}\big\rangle\big|z\text{-up}\big\rangle$

**(c)** or, in a random order, 5,000 pairs in $\big|x\text{-up}\big\rangle\big|x\text{-down}\big\rangle$ and 5,000 in $\big|x\text{-down}\big\rangle\big|x\text{-up}\big\rangle$.

Alice and Bob are far from each other, and each receives one particle of every pair. By carrying out (local) Stern-Gerlach experiments on their particles and comparing their results afterwards, how can they decide whether the source was of type (a), (b), or (c)?

**Exercise 39: Allcock's paradox**[1]
Allcock considered a "soft detector," i.e., one for which the particle may fly through the detector volume for a while before being detected. An as effective description, Allcock proposed an imaginary potential. For example, for a single particle in 1d and a detector in the right half axis, he considered the Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} - iv1_{x\geq 0}\psi \tag{2}$$

with $v > 0$ a constant. The time evolution then is not unitary.

[1] G.R. Allcock: The time of arrival in quantum mechanics II. The individual measurement. *Annals of Physics* **53**: 286–310 (1969)

**(a)** Derive from (1) the continuity equation

$$\frac{\partial \rho}{\partial t} = -\operatorname{div} j - \tfrac{2v}{\hbar} 1_{x \geq 0} \rho \tag{3}$$

for $\rho = |\psi|^2$ and $j = \frac{\hbar}{m}\mathrm{Im}[\psi^* \partial\psi/\partial x]$. Eq. (2) is the evolution of the probability density of a particle that moves along Bohmian trajectories and disappears spontaneously (stochastically) at rate $2v/\hbar$ whenever it stays in the region $x \geq 0$. $\|\psi_t\|^2$ is a decreasing function of $t$ and represents the probability that the particle has not been detected (and disappeared from the model) yet.

**(b)** To obain an effective description of a hard detector (i.e., one that will detect the particle as soon as it reaches the region $x \geq 0$), Allcock assumed that $\psi_0$ is concentrated in $\{x < 0\}$ and took the limit $v \to \infty$, but found that the particle never gets detected ($\|\psi_t\|^2 = \text{const.}$)! That is parallel to the quantum Zeno effect.

Prove the following simplified version: In a 2d Hilbert space $\mathbb{C}^2$, let $\psi_0 = (1, 0)$ evolve with the (non-self-adjoint) Hamiltonian

$$H_v = \begin{pmatrix} 0 & 1 \\ 1 & -iv \end{pmatrix} . \tag{4}$$

Then for every $t > 0$, $\psi_t = e^{-iH_v t/\hbar}\psi_0 \to \psi_0$ as $v \to \infty$.

**Hand in:** by Tuesday January 18, 2022, 8:15am via `urm.math.uni-tuebingen.de`

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**Reading assignment** due Tuesday January 18, 2022:
A. Einstein, B. Podolsky, N. Rosen: Can Quantum-Mechanical Description of Physical Reality Be Considered Complete? *Physical Review* **47**: 777–780 (1935)