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Winter semester 2021/22
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Foundations of Quantum Mechanics: Assignment 10

Exercise 40: Essay question. Describe the Einstein–Podolsky–Rosen argument (either in terms of position and momentum or in terms of spin matrices).

Exercise 41: Positive operators
An operator $S : \mathbb{C}^d \to \mathbb{C}^d$ is positive (= positive semi-definite) iff $\langle\psi|S\psi\rangle \geq 0$ for all ψ. Are the following statements about operators on $\mathbb{C}^d$ true or false? Justify your answers.

a) $R^\dagger R$ is always a positive operator.

b) If E is a positive operator, then so is $R^\dagger E R$.

c) The positive operators form a subspace of the space of self-adjoint operators.

d) The sum of two projections is positive only if they commute.

e) e^{At} is a positive operator for every self-adjoint A and $t \in \mathbb{R}$.

Exercise 42: Symmetrizer and anti-symmetrizer
Let S_N be the group of permutations of $\{1, \ldots, N\}$. A function $\psi(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N)$ is called

$$\textit{symmetric} \text{ iff } \psi(\boldsymbol{x}_{\pi(1)}, \ldots, \boldsymbol{x}_{\pi(N)}) = \psi(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) \tag{1}$$

$$\textit{anti-symmetric} \text{ iff } \psi(\boldsymbol{x}_{\pi(1)}, \ldots, \boldsymbol{x}_{\pi(N)}) = (-)^\pi \psi(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N)\,, \tag{2}$$

for every $\pi \in S_N$, where $(-)^\pi$ denotes the sign of π (i.e., $+1$ if π is even, -1 if odd). Let $\mathscr{S}_+$ denote the subspace of all symmetric functions in $L^2(\mathbb{R}^{3N})$ and $\mathscr{S}_-$ that of anti-symmetric functions. Show that the operators $P_\pm$ defined by

$$P_+\psi(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) = \frac{1}{N!} \sum_{\pi \in S_N} \psi(\boldsymbol{x}_{\pi(1)}, \ldots, \boldsymbol{x}_{\pi(N)}) \tag{3}$$

$$P_-\psi(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) = \frac{1}{N!} \sum_{\pi \in S_N} (-)^\pi \psi(\boldsymbol{x}_{\pi(1)}, \ldots, \boldsymbol{x}_{\pi(N)}) \tag{4}$$

are the projections to $\mathscr{S}_\pm$. Use without proof that $(-)^\pi(-)^\rho = (-)^{\pi\circ\rho}$ for $\pi, \rho \in S_N$.

Exercise 43: Sum of projections
Let $\mathscr{H}$ be a Hilbert space of finite dimension, let P_1 and P_2 be projections in $\mathscr{H}$, $P_i = P_i^\dagger$ and $P_i^2 = P_i$, and let $\mathscr{H}_i$ be the range of P_i. Show that if $Q := P_1 + P_2$ is also a projection ($Q = Q^\dagger$ and $Q^2 = Q$), then **(a)** $\mathscr{H}_1 \perp \mathscr{H}_2$, and **(b)** the range $\mathscr{K}$ of Q is the span of $\mathscr{H}_1 \cup \mathscr{H}_2$.

Hand in: by Tuesday January 25, 2022, 8:15am via `urm.math.uni-tuebingen.de`

Reading assignment due Thursday January 27, 2022:
N. Bohr: Can Quantum-Mechanical Description of Physical Reality Be Considered Complete? *Physical Review* **48**: 696–702 (1935)