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Winter semester 2021/22
January 25, 2022

Foundations of Quantum Mechanics: Assignment 11

Exercise 43: Essay question. Describe Einstein's boxes argument.

Exercise 44: Can't distinguish non-orthogonal state vectors with POVMs
In Exercise 22(b) in Assignment 5, it was shown that Bob, when allowed to use a quantum measurement of *any self-adjoint operator* on a given particle, is unable to decide with certainty whether the quantum state was $(1,0)$ or $\frac{1}{\sqrt{2}}(1,1)$. What if Bob is allowed to use *any experiment whatsoever*? Use the main theorem about POVMs.

Exercise 45: POVMs
(a) Suppose E_1 and E_2 are POVMs on $\mathscr{Z}_1$ and $\mathscr{Z}_2$, respectively, both acting on $\mathscr{H}$; let $q_1, q_2 \in [0,1]$ with $q_1 + q_2 = 1$. Show that $E(B) := q_1 E_1(B \cap \mathscr{Z}_1) + q_2 E_2(B \cap \mathscr{Z}_2)$ defines a POVM on $\mathscr{Z}_1 \cup \mathscr{Z}_2$.

(b) Suppose experiment $\mathscr{E}_1$ has distribution of outcomes $\langle\psi|E_1(\cdot)|\psi\rangle$, and $\mathscr{E}_2$ has distribution of outcomes $\langle\psi|E_2(\cdot)|\psi\rangle$. Describe an experiment with distribution of outcomes $\langle\psi|E(\cdot)|\psi\rangle$.

(c) Give an example of a POVM for which the E_z do not pairwise commute. *Suggestion*: Choose $E_1(z)$ that does not commute with $E_2(z')$ for $\mathscr{Z}_1 \cap \mathscr{Z}_2 = \emptyset$.

Exercise 46: Main theorem about POVMs
The proof of the main theorem from Bohmian mechanics assumes that at the initial time t_i of the experiment, the joint wave function factorizes, $\Psi_{t_i} = \psi \otimes \phi$. What if factorization is not exactly satisfied, but only approximately? Then the probability distribution of the outcome Z is still approximately given by $\langle\psi|E(\cdot)|\psi\rangle$. To make this statement precise, suppose that

$$\Psi_{t_i} = c\psi \otimes \phi + \Delta\Psi\,, \tag{1}$$

where $\|\Delta\Psi\| \ll 1$, $\|\psi\| = \|\phi\| = 1$, and $c = \sqrt{1 - \|\Delta\Psi\|^2}$ (which is close to 1). Use the Cauchy-Schwarz inequality,

$$\big|\langle f|g\rangle\big| \leq \|f\|\,\|g\|\,, \tag{2}$$

to show that, for any $B \subseteq \mathscr{Z}$,

$$\Big|\mathbb{P}(Z \in B) - \langle\psi|E(B)|\psi\rangle\Big| < 3\|\Delta\Psi\|\,. \tag{3}$$

Hand in: by Tuesday February 1, 2022, 8:15am via `urm.math.uni-tuebingen.de`

Reading assignment due Thursday February 3, 2022:
J. Bell: Bertlmann's Socks and the Nature of Reality. *Journal de Physique* **42**: C2 41–61 (1981)