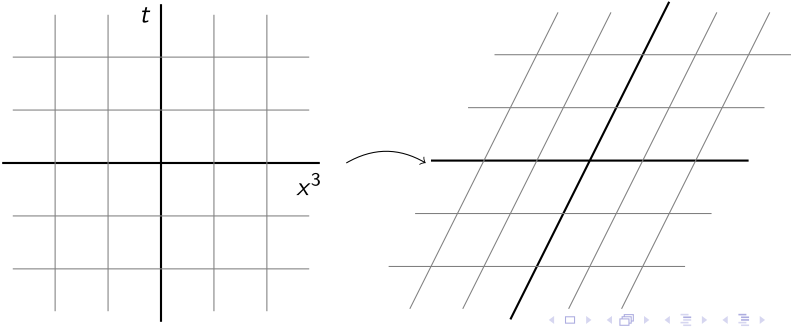
Galilean relativity

- space-time point $(t, \boldsymbol{x})$
- Galilean boost

$$
\begin{aligned}
t' &= t \\
x'_1 &= x_1 + v_1 t \\
x'_2 &= x_2 + v_2 t \\
x'_3 &= x_3 + v_3 t
\end{aligned}
$$



Galilean relativity

- Newtonian is invariant under Galilean transformations.
- That is, transformation of a solution yields another solution.
- It is in principle impossible to determine empirically how fast any object (such as the Earth) is moving.
- Two classic views:

Newton

Space is at rest. Therefore, there is a fact about the actual velocity of an object, even though this velocity cannot be measured empirically.

Leibniz

There are no facts in nature about the absolute velocity of an object, only about relative velocities. There is no fact about the identity of space points at different times.

Leibniz' view

Space-time as a 4-dimensional real affine space A equipped with an equivalence class of affine-linear functions $t : A \to \mathbb{R}$ ("time"), where equivalence means that two functions differ by addition of a real constant (so there is no fact about which time is time 0), and the structure of a Euclidean space on each level set of t ("time slice") such that every translation of A is an isometry on each time slice.

Einstein's special relativity

In contrast to Newtonian mechanics, Maxwell's equations, the fundamental equations governing the electromagnetic field according to classical electrodynamics, are not invariant under Galilean transformations. However, they are invariant under a similar family of linear transformations $\mathbb{R}^4 \to \mathbb{R}^4$, the Lorentz transformations.

Einstein's principle of relativity (1905)

The true symmetry of space-time is given by the Lorentz transformations. All laws of nature are invariant under Lorentz transformations.

Minkowski space

Definition

Minkowski space is a 4d real vector space M equipped with a symmetric bilinear form $(\cdot,\cdot) : M \times M \to \mathbb{R}$ (called *metric*) which in a suitable basis b_0, b_1, b_2, b_3 is given by

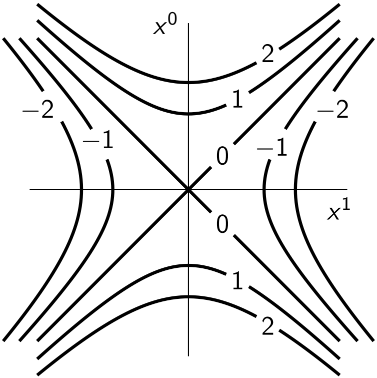
$$(x,y) = x^0y^0 - x^1y^1 - x^2y^2 - x^3y^3 = \sum_{\mu,\nu=0}^{3} x^\mu y^\nu \eta_{\mu\nu}$$

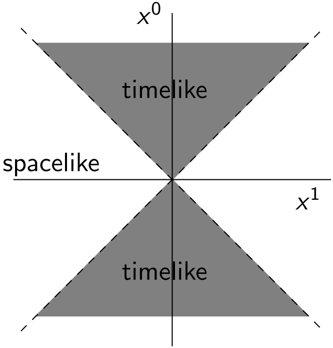
$$\text{with } \eta = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}.$$

Such a basis is called a Lorentz frame.

indefinite; $x^0 = ct$ with $c =$ speed of light; we write x^μ for x; Einstein sum convention

$$x^\mu y^\nu \eta_{\mu\nu} \text{ instead of } \sum_{\mu,\nu=0}^{3} x^\mu y^\nu \eta_{\mu\nu}\,.$$


x^0
x^1
2
1
0
−1
−2


x^0
x^1
timelike
spacelike
timelike

Lorentz transformations

Definition

A Lorentz transformation is a linear mapping $\Lambda : M \to M$ that preserves the metric,

$$(\Lambda x, \Lambda y) = (x, y) \quad \forall x, y \in M\,.$$

Alternatively, it is the coordinate expression of such a mapping relative to a Lorentz frame, i.e., a matrix $\Lambda^{\mu}{}_{\nu}$ such that

$$\Lambda^{\mu}{}_{\lambda} \eta_{\mu\nu} \Lambda^{\nu}{}_{\rho} = \eta_{\lambda\rho} \text{ or } \Lambda^{t} \eta \Lambda = \eta\,.$$

These matrices form a group called the Lorentz group or $O(1,3)$.

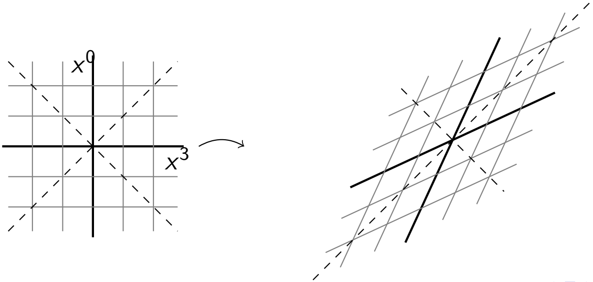
Lorentz transformations

Example

The Lorentz boost in the x^3 direction with rapidity ξ is the matrix

$$(\Lambda^\mu{}_\nu) = \begin{pmatrix} \cosh\xi & & & \sinh\xi \\ & 1 & & \\ & & 1 & \\ \sinh\xi & & & \cosh\xi \end{pmatrix}.$$

When composing Lorentz boosts in x^3 direction, rapidities add.



Consequences

- Analog of Newton's view: space is at rest, but we cannot find out which frame that is.
- Analog of Leibniz's view: there is no absolute rest frame.
- Different Lorentz frame disagree about which space-time points are simultaneous.
- Minkowski space provides no fact about which space-time points really are simultaneous, or about the temporal order of spacelike separated points.
- Space and time as such do not exist; only space-time exists, and it is an affine Minkowski space.