eq. of motion, $i = 1 \dots N$

gravitational mass

$$\underset{\text{inertial}}{m_i} \frac{d^2 \underline{Q}_i(t)}{dt^2} = \underbrace{- \sum_{j \neq i} \frac{e_i e_j}{4\pi\varepsilon_0} \frac{\underline{Q}_j(t) - \underline{Q}_i(t)}{|\underline{Q}_j - \underline{Q}_i|^3}}_{\text{Coulomb force}} + \underbrace{\sum_{j \neq i} G m_i m_j \frac{\underline{Q}_j - \underline{Q}_i}{|\underline{Q}_j - \underline{Q}_i|^3}}_{\text{gravitational force}}$$

mass · acc = force

$i \longrightarrow j$

reductionism = behavior of large objects follows from behavior of micro objects.

deterministic

baseball

$$\underset{\substack{\uparrow\\ \text{total mass}}}{M} \frac{d^2\underline{Q}(t)}{dt^2} = \underbrace{-\overset{\substack{\text{friction coeff.}\\ \downarrow}}{\gamma} \frac{d\underline{Q}}{dt}}_{\text{friction force}} \underbrace{- M \overset{\substack{\text{field strength}\\ \downarrow}}{g} \begin{pmatrix}0\\0\\1\end{pmatrix}}_{\text{gravitational force of Earth}},$$

$g = 9.81\ m/s^2$

effective eq. of motion (approx.)

## Time reversal

time reversal invariance = reversibility

**Prop** If $t \mapsto q(t) = (\underline{Q}_1(t), \dots, \underline{Q}_N(t))$ is a sol. of eq. of motion, then so is $t \mapsto q(-t)$.

D. Bernoulli, L. Boltzmann.

macroscopic irreversibility

## Conservation laws

**Def** Let $\underline{v}_i(t) = \dfrac{d\,\underline{Q}_i(t)}{dt}$

energy $E(t) = \underbrace{\sum_{k=1}^{N} \frac{m_k}{2} \underline{v}_k^2}_{\text{kinetic energy}} + \overbrace{\sum_{\substack{j,k=1 \\ j<k}}^{N} \left( G m_j m_k - \frac{e_j e_k}{4\pi\varepsilon_0} \right) \frac{1}{|\underline{Q}_j - \underline{Q}_k|}}^{\text{potential energy}}$

momentum $\underline{p}(t) = \sum_{k=1}^{N} m_k \underline{v}_k$

cross product $\wedge$

angular momentum $\underline{L}(t) = \sum_{k=1}^{N} m_k \, \underline{Q}_k \times \underline{v}_k$

$\underline{u} \times \underline{v} = \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{pmatrix}$. $\underline{v}^2 = \underline{v} \cdot \underline{v} = |\underline{v}|^2$.

**Prop** $E, \underline{p}, \underline{L}$ are conserved.

## Hamiltonian systems

dynamical system = ODE : $\frac{dX(t)}{dt} = F(X,t)$

$X(t) \in \mathbb{R}^n$, $F : \Omega \to \mathbb{R}^n$ given

$\Omega \subseteq \mathbb{R}^n \times \mathbb{R}$

Newt. Mech. is a dynamical system, $n = 6N$,

$\Omega = \{ (\underline{Q}_1, \dots, \underline{Q}_N, \underline{v}_1, \dots, \underline{v}_N) \in \mathbb{R}^{6N} : \underline{Q}_i \neq \underline{Q}_j \ \forall i \neq j \} \times \mathbb{R}_t$.

$T_t(X(0)) = X(t)$, $T_t$ : flow map

$T_s T_t = T_{s+t}$ and $T_0 = id_{\text{phase space}}$.

Hamiltonian systems $\subseteq$ dyn. systems

$\mathbb{R}^n$, $n=2r$, $x=(q_1 \cdots q_r, p_1 \cdots p_r)$, ODE

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}$$

$H : \mathbb{R}^n \to \mathbb{R}$

Hamiltonian fct

= Hamiltonian

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}$$

Newtonian mech: $r=3N$, $q=(\underline{Q}_1, \cdots, \underline{Q}_N)$,

$p=(\underline{p}_1, \cdots, \underline{p}_N)$, $\underline{p}_k = m_k \underline{v}_k$, $H(q,p) = \sum_{k=1}^{N} \frac{\underline{p}_k^2}{2m_k} + V$

$(\underline{p}_k \longleftrightarrow -i\hbar \nabla_k)$

# Hamiltonian systems on manifolds

$\dim M = 2r = n$ (replaces $\mathbb{R}^n$)

symplectic form $\omega$ = non-degenerate differential 2-form with $d\omega = 0$.

exterior derivative

eq. of motion in $M$:

$$\omega\Big(\underbrace{\frac{dX(t)}{dt}}, \cdot\Big) = dH \qquad H : M \to \mathbb{R}$$

(In $M = \mathbb{R}^n$: $\omega = \left(\begin{array}{c|c} 0 & I \\ \hline -I & 0 \end{array}\right)$, $\omega\left(\frac{dX}{dt}, \cdot\right) = \begin{pmatrix} \frac{dp_i}{dt} \\ \frac{dq_i}{dt} \end{pmatrix}$
$dH = \operatorname{grad} H$.)