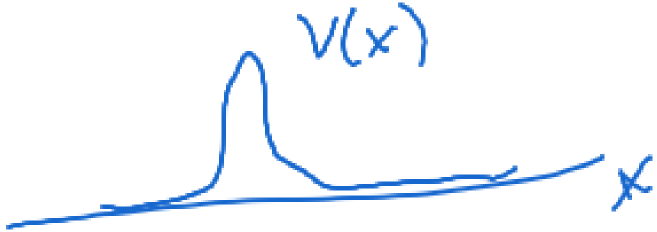
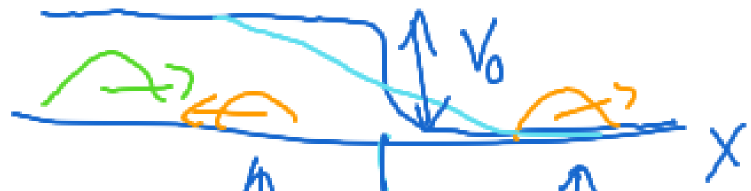
Tunneling



anti-tunneling

paradoxical reflection



downward potential step

$\rho_R = \|\psi_R\|^2$ $\|\psi_T\|^2$

$V(x) = V_0 \, 1_{x<0}$

$\lim_{V_0 \to \infty} \rho_R = 1.$

Heisenberg Uncertainty Relation

$\langle X \rangle$ = expectation of $X = \mathbb{E}(X)$.

variance$(X) = \langle (X - \langle X \rangle)^2 \rangle = \sigma^2$

X = outcome of momentum measurement

$\psi \in L^2(\mathbb{R}^1)$

$$\sigma_p^2 = \langle (p - \langle p \rangle)^2 \rangle$$

$$P = -i\hbar \frac{\partial}{\partial x}$$

$$= \langle p^2 - 2p\langle p \rangle + \langle p \rangle^2 \rangle$$

$$= \langle p^2 \rangle - 2\langle p \rangle \langle p \rangle + \langle p \rangle^2$$

$$= \langle p^2 \rangle - \langle p \rangle^2 = \langle \psi | P^2 \psi \rangle - \langle \psi | P \psi \rangle^2$$

$$= \langle \psi | (P - \underbrace{\langle \psi | P \psi \rangle}) ^2 \psi \rangle$$

$$\sigma_X^2 = \langle (Q(0) - \langle Q(0) \rangle)^2 \rangle$$

$$= \int dx \, |\psi(x)|^2 \, (x - \langle Q(0) \rangle)^2$$

$$= \langle x^2 \rangle - \langle x \rangle^2 \,, \qquad \left[\langle x^2 \rangle = \int x^2 \, |\psi(x)|^2 dx \right]$$

$$= \langle \psi | (X - \langle \psi | X \psi \rangle)^2 \psi \rangle \qquad \left[\begin{aligned} &= \langle \psi | X^2 \psi \rangle \\ \langle x \rangle &= \langle \psi | X \psi \rangle \end{aligned} \right]$$

$$= \langle \psi | X^2 \psi \rangle - \langle \psi | X \psi \rangle^2$$

$$X \psi(x) = x \, \psi(x)$$

Thm (Heisenberg uncertainty relation)

$\forall \psi \in L^2(\mathbb{R}^1)$ with $\|\psi\| = 1$,

$$\sigma_x \sigma_p \geq \frac{\hbar}{2}.$$

Pf later (general form).

Rem $\sigma_p^2 = \|P\psi\|^2 - \langle \psi | P\psi \rangle^2 \begin{cases} < \infty & \text{for } \psi \in \operatorname{domain}(P) \\ = \infty & \text{otherwise.} \end{cases}$

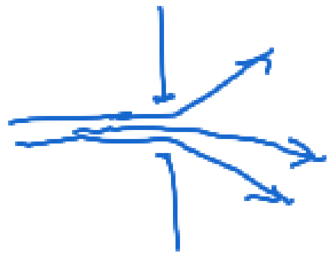
if $\psi \notin \operatorname{dom}(P)$

lhs $= \infty$ b/c $\sigma_x > 0$ (ineq. trivially true)

Ex Gaussian $\sigma_p = \frac{\hbar}{2\sigma_x}$, so $\sigma_x \sigma_p = \frac{\hbar}{2}$.

Ex slit

In BM



1) if traj. are close, they move apart.

2) we know $Q \sim |\varphi(x)|^2$

When we know Q with small inaccuracy, then we know $\frac{dQ}{dt}$ or v with large inaccuracy

(Q, ψ)

limitation to knowledge

In OQM

$\nexists$ traj., ψ is complete description of state

Heisenberg UR $\neq$ limitation to knowledge

Self-adjoint operators

Certain experiments are called "quantum measurements" of "observables".
Every observable is associated with a self-adjoint operator.

Thm (adjoint) Every bdd. op. $A : \mathcal{H} \to \mathcal{H}$ possesses one and only one adjoint. op. $A^\dagger : \mathcal{H} \to \mathcal{H}$, defined by

$$\langle \psi | A \phi \rangle = \langle A^\dagger \psi | \phi \rangle \quad (*) \quad \forall \psi, \phi \in \mathcal{H}.$$

For unbdd. $A : \mathcal{D}(A) \to \mathcal{H}$, $A^\dagger$ is def'd by $(*)$ $\forall \psi \in \mathcal{D}(A^\dagger) \, \forall \phi \in \mathcal{D}(A)$, where $\mathcal{D}(A^\dagger) = \{ \psi \in \mathcal{H} : \exists \chi \in \mathcal{H} \, \forall \phi \in \mathcal{D}(A) : \langle \psi | A \phi \rangle = \langle \chi | \phi \rangle \}$.

<u>Def</u> A self-adj. $\Longleftrightarrow$ $A = A^\dagger$.

Then $\underline{\langle \psi | A\phi \rangle = \langle A\psi | \phi \rangle \quad \forall \psi, \phi \in \mathcal{D}(A)}$.

<u>Ex</u> • $\mathcal{H} = \mathbb{C}^n$, every op. A is bdd., corresponds to a matrix A_{ij}. The matrix of $A^\dagger$ is

$$(A^\dagger)_{ij} = (A_{ji})^*$$ "adjoint matrix is the conjugate transpose"

A self-adj iff $A_{ji}^* = A_{ij}$

• $\mathcal{H} = L^2(\mathbb{R}^d)$, multiplication op.

$f: \mathbb{R}^d \to \mathbb{C}$

$$A\,\psi(\underline{x}) = f(\underline{x})\psi(\underline{x})$$

$A^+ =$ mult. by f^*.

$A = A^+$ iff $f = f^*$ i.e. $f : \mathbb{R}^d \to \mathbb{R}$.

- $(AB)^+ = B^+ A^+$ and $\exp(A)^+ = \exp(A^+)$
- On $\mathcal{H} = L^2(\mathbb{R}^d)$, the momentum op

  $P_j = -i\hbar \frac{\partial}{\partial x_j}$ is self-adjoint on $\mathcal{D} =$ first Sobolev space.
- $H = \sum_{j=1}^{N} \frac{1}{2m_j} P_j^2 + V$ is self-adjoint.

Spectral theorem

$$A\psi = \alpha\psi$$

$\text{eigenspace}(\alpha) = \{\text{eigenvectors with eigenvalue } \alpha\} \cup \{0\}$.

α degenerate $\Longleftrightarrow$ $\dim \text{eigenspace}(\alpha) > 1$.

A self-adj $\Rightarrow$ $\alpha \in \mathbb{R}$, b/c ψ is e.v. with e.v. α,

$$\alpha\underbrace{\langle\psi|\psi\rangle}_{\neq 0} = \langle\psi|\alpha\psi\rangle = \langle\psi|A\psi\rangle = \langle A\psi|\psi\rangle = \langle\alpha\psi|\psi\rangle = \bar{\alpha}\underbrace{\langle\psi|\psi\rangle}_{\neq 0}$$

Spectral thm For every self-adj. A there is a (generalized) orthonormal basis $\{\phi_{\alpha,\lambda}\}$ consisting of (generalized) eigenvectors of A,

$$A\phi_{\alpha\lambda} = \alpha\phi_{\alpha\lambda} \qquad (\text{"eigenbasis of } A\text{"}).$$

Def ONB $= \{\phi_n\} \subset \mathcal{H}$ such that $\langle \phi_n | \phi_m \rangle = \delta_{nm}$

and $\forall \psi \in \mathcal{H}$: $\psi = \sum_n c_n \phi_n$

(Schauder basis vs. Hamel basis)

generalized ONB: (idea: $\psi = \int dk\, c_k\, \phi_k$)

unitary isomorphism $U : \mathcal{H} \to L^2(\Omega)$

$$(U\psi)(k) \simeq c_k \qquad (\text{e.g. } U = \mathcal{F})$$

ONB: $U : \mathcal{H} \to \ell^2 = L^2(\mathbb{N})$

$$\sum c_n \phi_n \longmapsto (c_1, c_2, c_3, \dots)$$

how to use the spectral thm:

$$\psi = \sum_{\alpha\lambda} c_{\alpha\lambda}\, \phi_{\alpha\lambda}$$

$$\Rightarrow \qquad A\psi = \sum_{\alpha\lambda} \alpha\, c_{\alpha\lambda}\, \phi_{\alpha\lambda}$$

GONB $\{\phi_{\alpha\lambda}\}$ diagonalizes A

Born's rule for self-adj. A In a quantum

measurement of A, the outcome is random with

prob. dist

$$\rho_A(\alpha) = \sum_{\lambda} \Big|\underbrace{\langle \phi_{\alpha\lambda} | \psi \rangle}_{c_{\alpha\lambda}}\Big|^2 = \sum_{\lambda} \big|(U\psi)(\alpha,\lambda)\big|^2$$