__Born's rule__ $\quad A$ self-adj. op. in $\mathcal{H}$.

$$\rho_A(\alpha) = \sum_\lambda |\langle \phi_{\alpha,\lambda} | \psi \rangle|^2 = \underline{\sum_\lambda |(U\psi)(\alpha,\lambda)|^2}$$

$$U : \mathcal{H} \to L^2(\Omega)$$

__Conseq__ $\quad \mathbb{E}(\alpha) = \int d\alpha \, \alpha \, \rho_A(\alpha)$

$$= \int_\Omega d(\alpha,\lambda) \, \alpha \, \cancel{\sum_\lambda} |U\psi(\alpha,\lambda)|^2$$

$$= \langle U\psi | U U^{-1} M \, U\psi \rangle_\Omega$$

$$= \langle \psi | \underbrace{U^{-1} M U}_{A} \psi \rangle_{\mathcal{H}} = \langle \psi | A \psi \rangle$$

[M = mult. by $\alpha : \Omega \to \mathbb{R}$

$M : \underbrace{L^2(\Omega)}_{\text{or dense subspace therein}} \to L^2(\Omega)$]

Apply spectral thm to H

$$H\,\phi_{E,\lambda} = E\,\phi_{E,\lambda}\,, \qquad \{\phi_{E,\lambda}\} \text{ GONB}$$

energy eigenfcts

$$\mathcal{H} \ni \psi = \sum_{E,\lambda} c_{E,\lambda}\,\phi_{E,\lambda}$$

$$\psi_t = \underbrace{e^{-iHt/\hbar}}\,\psi = \sum_{E,\lambda} \underbrace{e^{-iEt/\hbar}}\, c_{E,\lambda}\,\phi_{E,\lambda}$$

i.e., $\underline{c_{E,\lambda}(t) = e^{-iEt/\hbar}\, c_{E,\lambda}(0)}$

Conservation laws in QM

$E(t)$, $\underline{p}(t)$, $\underline{L}(t)$ t-indep. in Newtonian mech.

no analog in QM.

$|c_{E,\lambda}(t)|^2$ are t-indep.: conservation law,

no analog in Newtonian mech.

H, P, L op.s are conserved:

ONB $\{\phi_i\}$, $\{\phi_i(t)\}$ ONB, $\underline{S_{ij}^{(t)} = \langle \phi_i^{(t)} | S \phi_j^{(t)} \rangle}$ t-indep.

if $\underline{e^{iHt} S e^{-iHt} = S}$ $\Longleftarrow$ $SH = HS$

"S is conserved"

$PH_0 = H_0 P$

$PH \neq HP$

Conseq for actual values

1) quantum measurement of S

no conservation of S.

2) Ex tunneling, $H = H_0 + V$

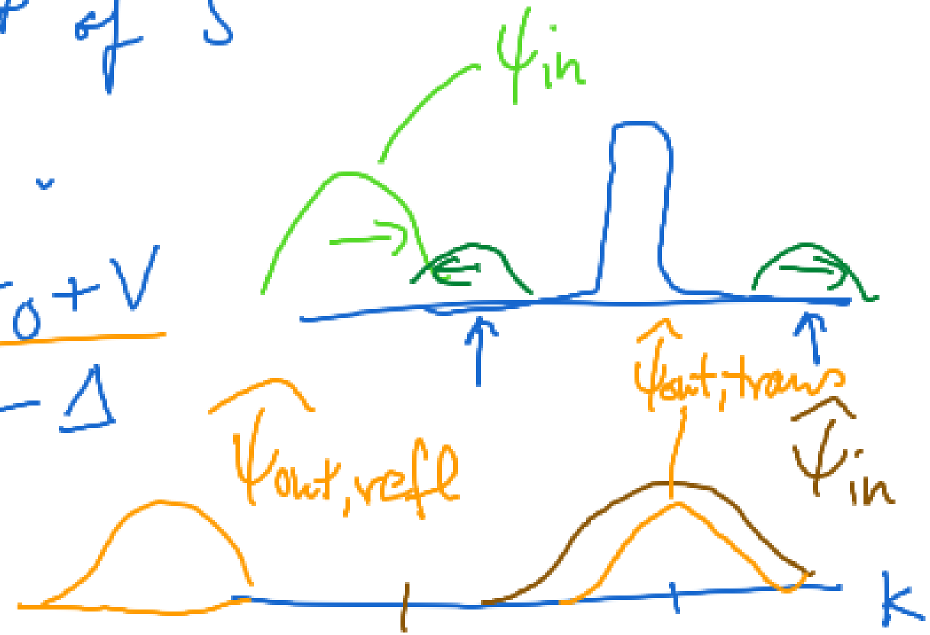
early: $\rho_H(E) \approx \rho_{H_0}(E)$ $\quad H_0 = -\Delta$

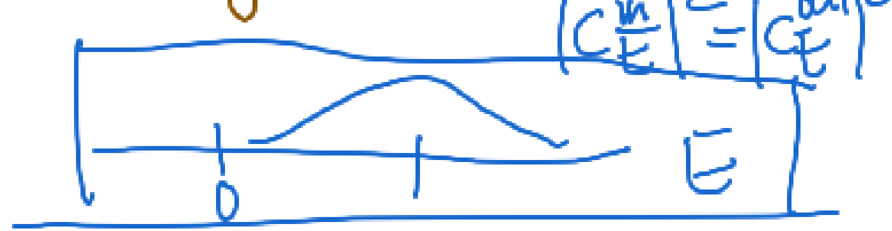
late: $\rho_H(E) \approx \rho_{H_0}(E)$

$\rho_H(E,t)$ is t-indep.

$\rho_H(E, \text{early}) = \rho_H(E, \text{late})$

$\rho_{H_0}(E, \text{early}) \approx \rho_{H_0}(E, \text{late})$





Spin

$\psi : \mathbb{R}^3 \longrightarrow \mathbb{C}$ before

$\psi : \underline{\mathbb{R}^3} \longrightarrow \underline{\mathbb{C}^2}$ now

$\mathbb{C}^2 =$ spin space $= S$

$S \ni \phi$ ~~spin vector~~ spinor

Spinors

S is a Hilbert space, $\dim S = 2$.

$S \ni \phi$ is associated with a vector in physical space $\mathbb{R}^3$.

$$\underline{\omega} : S \longrightarrow \mathbb{R}^3$$

$$\underline{\omega}(\phi) = \left(\sum_{r,s=1}^{2} \phi_r^* (\sigma_1)_{rs} \phi_s , \ \ldots \sigma_2 \ldots , \ \ldots \sigma_3 \ldots \right)$$

$$\omega_a(\phi) = \langle \phi | \sigma_a \phi \rangle_S$$

$$\langle \phi | \chi \rangle = \sum_{s=1}^{2} \phi_s^* \chi_s = \phi^* \chi = \phi^+ \chi$$

Pauli matrices: $\underline{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

self-adj, complex 2×2, have e.v.s ± 1,

$\underline{\text{Ex}}$ $\phi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\underline{w}(\phi) = \left(\phi^* \sigma_1 \phi, \phi^* \sigma_2 \phi, \phi^* \sigma_3 \phi \right)$

$\phi^* M \phi = M_{11}$ $\quad = (0, 0, 1)$ "up spinor"

$\phi = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\underline{w}(\phi) = (0, 0, -1)$ "down spinor"

$\phi^* M \phi = M_{22}$

Properties

- $\underline{\omega}(z\phi) = |z|^2 \underline{\omega}(\phi)$ $\quad z \in \mathbb{C}$

- $\|\underline{\omega}(\phi)\|_{\mathbb{R}^3} = \|\phi\|_S^2 = \phi^* \phi$ (exercise)

- Def angle in $\mathcal{H}$: $\cos\theta = \dfrac{|\langle\phi|\chi\rangle|}{\|\phi\|\,\|\chi\|}$

Thm Suppose $\phi, \chi \in S$, $\|\phi\|_S = 1 = \|\chi\|_S$ with angle θ. Then $\text{angle}_{\mathbb{R}^3}(\underline{\omega}(\phi), \underline{\omega}(\chi)) = 2\theta$.

Spin-$\frac{1}{2}$. Ex $\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$ have $\theta = 90°$

$\underline{\omega}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix}, \underline{\omega}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\-1\end{pmatrix}$ have angle $180°$.

Other spins

spinors of spin-$\frac{1}{2}$, spin-1, spin-$\frac{3}{2}$, spin 2, spin $\frac{5}{2}$, ...

The space of spin-s spinors has dim $2s+1$.

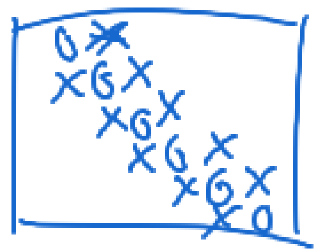
spin-0: $\dim = 1$, $\psi : \mathbb{R}^3 \to \mathbb{C}$

spin-s:

$$(\sigma_1)_{rr'} = \frac{1}{2}\left(\delta_{r,r'+1} + \delta_{r+1,r'}\right)\sqrt{(2s+1)(r+r'-1) - rr'}\,, \quad r, r' \in \{1 .. 2s+1\}$$

$$(\sigma_2)_{rr'} = \frac{-i}{2}\left(\underline{\delta_{r,r'+1}} - \delta_{r+1,r'}\right)\sqrt{(2s+1)(r+r'-1) - rr'}$$

$$(\sigma_3)_{rr'} = \underline{\delta_{rr'}}\,(2s+1-r)$$



in nature, electrons have spin $\frac{1}{2}$

quarks have spin $\frac{1}{2}$

all matter particles have spin $\frac{1}{2}$.

force particles have integer spin either 1 or 2

photons 1

gluons 1

gravitons 2

except Higgs boson 0

(confirmed in 2012)