Spin

S, $\underline{w} : S \to \mathbb{R}^3$, $\underline{w}(\phi) = \phi^* \underline{\sigma}\, \phi$

The Pauli Eq. (Schr. eq. for spin-$\frac{1}{2}$)

$\psi(t, \underline{x}) \in S$

$$\begin{aligned} &(i\nabla - A)(i\nabla - A) \\ &= -\nabla^2 - iA\nabla - i\nabla A + A^2 \\ &\quad\; 2 \end{aligned}$$

$$\begin{bmatrix} B_3 & B_1 + iB_2 \\ B_1 - iB_2 & -B_3 \end{bmatrix}$$

$$i\hbar \frac{\partial \psi}{\partial t} = \frac{1}{2m}\left(-i\hbar\nabla - \underline{A}(\underline{x})\right)^2 \psi(\underline{x}) - \frac{\hbar}{2m}\,\underline{\sigma}\cdot\underline{B}(\underline{x})\,\psi(\underline{x})$$

$$+ V(\underline{x})\,\psi(\underline{x})$$

with $\underline{B}$ = magnetic field, V = el. + grav. potential

$\underline{A}$ = magnetic vector potential

$\underline{A}$ def'd by

$$\underline{B} = \nabla \times \underline{A} = \begin{pmatrix} \partial_2 A_3 - \partial_3 A_2 \\ \partial_3 A_1 - \partial_1 A_3 \\ \partial_1 A_2 - \partial_2 A_2 \end{pmatrix} = \mathrm{curl}(A)$$

($\underline{A}$ not uniquely determined) ("gauge freedom")

Hilbert space $\mathcal{H} = \underline{L^2(\mathbb{R}^3, S)} = L^2(\mathbb{R}^3, \mathbb{C}^2)$,

inner product

$$\langle \psi | \phi \rangle = \int_{\mathbb{R}^3} d^3\underline{x}\, \psi^*(\underline{x})\, \phi_2(\underline{x})$$

$$= \int_{\mathbb{R}^3} d^3\underline{x} \sum_{s=1} \psi_s^*(\underline{x})\, \phi_s(\underline{x}).$$

Born rule (for position, given spin-$\frac{1}{2}$)

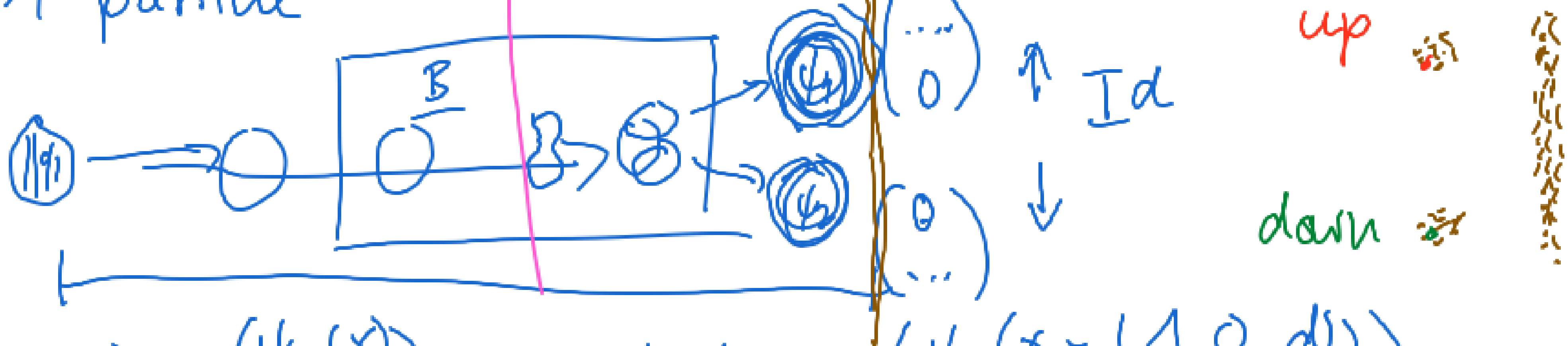
$$\rho(\underline{x}) = |\psi(\underline{x})|^2 = \psi^*(\underline{x})\psi(\underline{x}) = \sum_{s=1}^{2} |\psi_s(\underline{x})|^2.$$

$|\psi| =$ norm in S

$$X_j \psi(\underline{x}) = x_j \psi(\underline{x})$$

Stern-Gerlach experiment (1922 with silver atoms)

1 particle



$$\psi(\underline{x}) = \begin{pmatrix} \psi_1(\underline{x}) \\ \psi_2(\underline{x}) \end{pmatrix}, \quad \psi_t(\underline{x}) = \begin{pmatrix} \psi_1(\underline{x} - (1,0,d)) \\ \psi_2(\underline{x} - (1,0,-d)) \end{pmatrix}$$

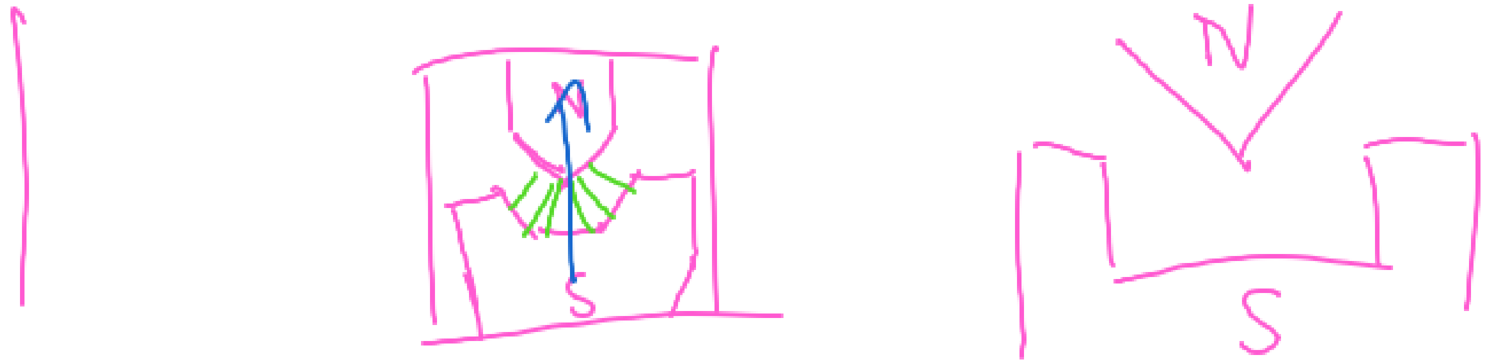
Simplest case: $\psi_S(\underline{x}) = \phi_S \, \underline{\chi(\underline{x})}$ disentangled

$$\Rightarrow \psi_t(\underline{x}) = \begin{pmatrix} \phi_1 \, \chi(\underline{x} - (1,0,d)) \\ \phi_2 \, \chi(\underline{x} - (1,0,-d)) \end{pmatrix}$$

$\overset{\text{Born}}{\Longrightarrow}$ $\mathrm{Prob}(\text{up}) = |\phi_1|^2 \, \overset{1}{\cancel{\|\chi\|^2}}$

$\mathrm{Prob}(\text{down}) = |\phi_2|^2$

"quantum measurement of σ_3"



SG exp. in y direction

$$\psi = \psi_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \psi_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \psi_+ \phi^{(+)} + \psi_- \phi^{(-)}$$

eigenbasis of σ_3 eigenbasis of σ_2

$\mathrm{Prob}_y(\text{up}) = |\psi_+|^2$, $\mathrm{Prob}_y(\text{down}) = |\psi_-|^2$

"quantum measurement of σ_2".

SG exp. in direction $\underline{n}$: $\underline{n} \cdot \underline{\sigma}$, eigenbasis $\phi^{(+)}_{\underline{n}}, \phi^{(-)}_{\underline{n}}$

Bohmian mechanics with spin

Bell (1966): 1 particle, $\psi : \mathbb{R}^4 \to S$ Pauli eq.

$\underline{Q}(t) \in \mathbb{R}^3$, $\quad \frac{d\underline{Q}}{dt} = \frac{\hbar}{m} \operatorname{Im} \frac{\psi^* \nabla \psi}{\psi^* \psi}(t, \underline{Q}(t))$.

(Alternative: + additional term)

$\underline{Q}(0) \sim |\psi_0(\underline{x})|^2 \xrightarrow{\text{equi.thm.}} \underline{Q}(t) \sim |\psi_t(\underline{x})|^2 \ \forall t$

uses cont. eq. $\frac{\partial |\psi(\underline{x})|^2}{\partial t} = -\nabla \cdot \Big(\underbrace{\frac{\hbar}{m} \operatorname{Im}(\psi^* \nabla \psi)}_{\text{current } \underline{j}} \Big)$

Conseq SG: makes correct predictions

Is the Electron a Spinning Ball?

Classically, charged spinning ball

$\Rightarrow$ magnetic moment $\underline{\mu}$

$\underline{\mu} \propto e\,\underline{\omega}$, $\quad \underline{\omega}$ = rotational velocity
direction = axis
magnitude = speed

Prediction: SG in $\underline{n}$, deflected by $\cancel{(\text{const.})}$ $\underline{n} \cdot \underline{\mu} = D$

$D \in [-|\underline{\mu}|, +|\underline{\mu}|]$

↯

Is there an actual spin vector?

Bohm, Schiller and Tiomno (1955)

$\underline{Q}(t)$, $\underline{S}(t)$ actual spin vector, $(\psi(t), \underline{Q}(t), \underline{S}(t))$

Proposal: $\underline{S}(t) = \dfrac{\underline{\omega}(\psi_t(\underline{Q}(t)))}{|\underline{\omega}(\psi_t(\underline{Q}(t)))|} = \dfrac{\psi^* \underline{\sigma} \psi}{\psi^* \psi}(t, \underline{Q}(t))$

2 problems: 1) SG "measurement of z-spin" outcome $\neq S_z(\tau_0)$

τ_0 = beginning of the SG experi.

2) outcome $= f(\underline{Q}(\tau_{final}))$, eq. of motion for $\underline{Q}(t)$ doesn't depend on $\underline{S}(t)$.

Upshot: $\underline{S}(t)$ is superfluous for SG

Many-Particle Systems

N particles $\quad \psi_{s_1 \ldots s_N}(\underline{x}_1, \ldots, \underline{x}_N)$, each $s_j \in \{1,2\}$

2^N components

$$\psi_t : \mathbb{R}^{3N} \to \mathbb{C}^{2^N} = (S)^{\otimes N} = S \otimes S \otimes \cdots \otimes S.$$

Pauli eq: $\quad i\hbar \frac{\partial \psi}{\partial t} = H^{(1)} \psi \qquad$ 1-particle

$$i\hbar \frac{\partial \psi}{\partial t} = \sum_{j=1}^{N} H_j^{(1)} \psi + V \psi \qquad \text{N particle}$$

Bohm-Bell eq. of motion: $\quad \frac{d\underline{Q}_j}{dt} = \frac{\hbar}{m_j} \operatorname{Im} \frac{\psi^* \nabla_j \psi}{\psi^* \psi}$

Born rule: $\rho(\underline{x}_1 \ldots \underline{x}_N) = \sum_{s_1 \ldots s_N = 1}^{2} |\psi_{s_1 \ldots s_N}(\underline{x}_1, \ldots, \underline{x}_N)|^2 = \psi^* \psi$