Proj. Postulate meas. A at t on ψ_{t-}, obtain α

then $\psi_{t+} = C \sum_{\lambda} |\phi_{\alpha\lambda}\rangle\langle\phi_{\alpha\lambda}|\psi_{t-}\rangle \leftarrow$

with $C = \left\| \sum_{\lambda} |\phi_{\alpha\lambda}\rangle\langle\phi_{\alpha\lambda}|\psi_{t-}\rangle \right\|^{-1}$

reduction of the wave packet = collapse of the wave function

Ex $A = \sigma_3$, $\psi_{t-} = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \in \underset{\sim}{S}$,

w/ prob $|\phi_1|^2$, outcome = "up", $\psi_{t+} = \begin{pmatrix} \phi_1/|\phi_1| \\ 0 \end{pmatrix}$

w/ prob $|\phi_2|^2$, down, $\psi_{t+} = \begin{pmatrix} 0 \\ \phi_2/|\phi_2| \end{pmatrix}$

Projections

$\{\phi_n\}$ ONB in $\mathcal{H}$, $\psi = \sum c_n \phi_n$,

so $c_n = \langle \phi_n | \psi \rangle$ b/c $\langle \phi_n | \sum_m c_m \phi_m \rangle =$

$$= \sum_m c_m \underbrace{\langle \phi_n | \phi_m \rangle}_{\delta_{nm}} = c_n .$$

$\tilde{\psi} = \sum_{n \in J} c_n \phi_n$, proj of ψ to $\overline{\text{span}}\{\phi_n \,|\, n \in J\}$

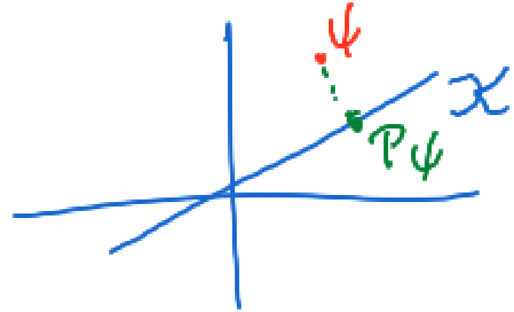
$$\text{matrix}(P) = \begin{bmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & & & & 0 & & & \\ & & & & & 0 & & \\ & & & & & & 0 & \\ & & & & & & & \ddots 0 \end{bmatrix}$$

proj. op. $\quad P = \sum_{n \in J} |\phi_n\rangle\langle\phi_n|$

Facts

- Any P is a proj. iff $P = P^\dagger$ and $P^2 = P$
- If $P = P^\dagger$ and $P^2 = P$ then e.v.s only 0 or 1.
- If $P = P^\dagger$ has e.v.s 0 or 1, then $P^2 = P$.

- subspace $\mathcal{K}$ is closed, $\mathcal{K} = \mathrm{range}(P)$
- $P|_{\mathcal{K}} = \mathrm{id}_{\mathcal{K}}$, $P|_{\mathcal{K}^\perp} = 0$
- $I - P$ = proj to $\mathcal{K}^\perp = \{\psi \in \mathcal{H} \mid \langle\psi|\phi\rangle = 0 \ \forall \phi \in \mathcal{K}\}$
- $P\psi$ is the el. of $\mathcal{K}$ closest to ψ
- If $\mathcal{K} = \mathbb{C}\psi$, then $P = |\psi\rangle\langle\psi|$
  and $\|\psi\| = 1$



So $\sum_\lambda |\phi_{\alpha\lambda}\rangle\langle\phi_{\alpha\lambda}|$ = proj. to eigenspace of A with e.v. α. $=: P_\alpha$

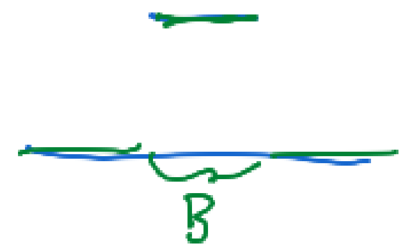
Thus, $\psi_{t+} = \dfrac{P_\alpha \psi_{t-}}{\|P_\alpha \psi_{t-}\|}$,

<u>Ex</u> $B \subseteq \mathbb{R}^{3N}$, $\mathcal{H} = L^2(\mathbb{R}^{3N}, \mathbb{C}^d)$

$$\mathcal{K}_B = \{\psi \in \mathcal{H} \mid \psi(x) = 0 \quad \forall x \notin B\} \quad \text{closed}$$

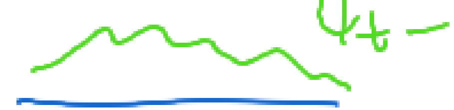
$$(P_B \psi)(x) = \begin{cases} \psi(x) & x \in B \\ 0 & x \notin B \end{cases}$$

i.e., P_B = mult. by $1_B(x)$



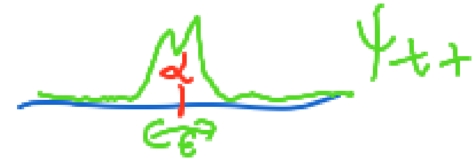
<u>Position Measurement</u> $\psi_{t-} \rightsquigarrow \delta(x-\alpha)$

α = outcome



realistic: inaccuracy ε

$$\psi_{t-} \rightsquigarrow C \, e^{-\frac{(x-\alpha)^2}{4\varepsilon^2}} \psi_{t-}(x)$$

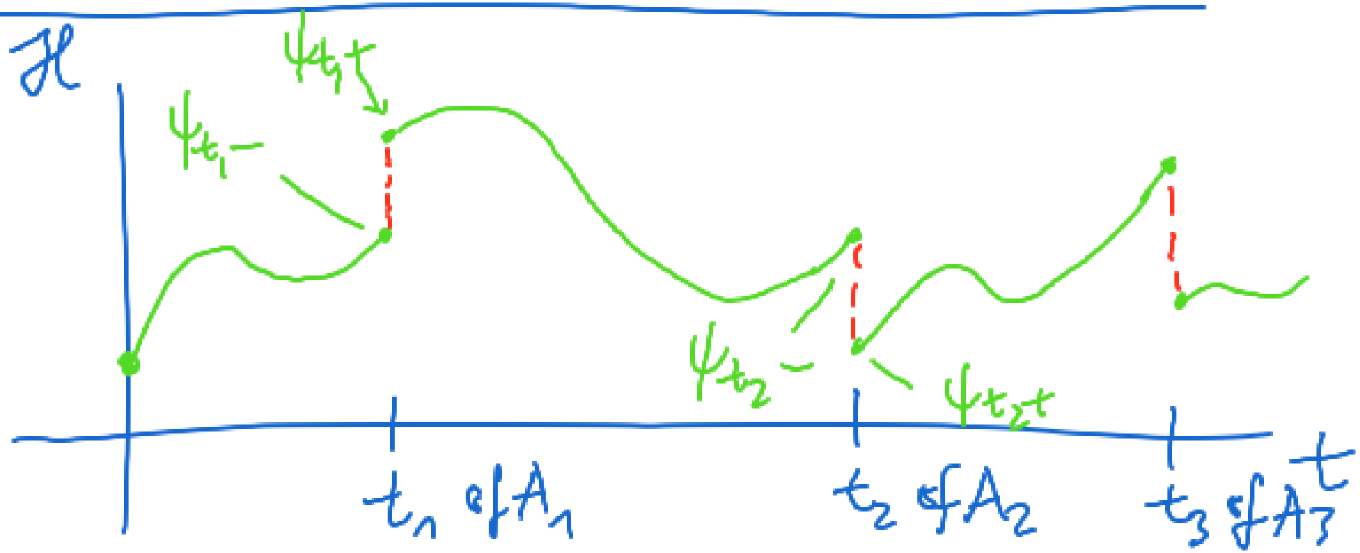


Another $B \subseteq \mathbb{R}^3$, $A = P_B$

$$\mathrm{Prob}(\alpha = 1) = \int_B dx \, |\psi_{t-}(x)|^2$$

If $\alpha = 1$, $\psi_{t-} \rightsquigarrow \psi_{t+} = \dfrac{P_B \psi_{t-}}{\| P_B \psi_{t-} \|}$

Consecutive Quantum Measurements



cond. prob.

$$P(E \mid F) = \frac{P(E \cap F)}{P(F)}$$

$$\Rightarrow \quad P(E \cap F) = P(E \mid F)\, P(F).$$

first $A = \sum_\alpha \alpha P_\alpha$, then $B = \sum_\beta \beta Q_\beta$

$$P(Z_A = \alpha) = \| P_\alpha \psi \|^2 \qquad (= P(F))$$

if $Z_A = \alpha$ then $\psi' = \dfrac{P_\alpha \psi}{\| P_\alpha \psi \|}$

$$P(Z_B = \beta \mid Z_A = \alpha) = \| Q_\beta \psi' \|^2 = \left\| Q_\beta \frac{P_\alpha \psi}{\| P_\alpha \psi \|} \right\|^2$$

$$(= P(E \mid F)) \qquad = \frac{1}{\| P_\alpha \psi \|^2} \| Q_\beta P_\alpha \psi \|^2$$

joint distr. $P(Z_A=\alpha, Z_B=\beta) = \frac{1}{\|P_\alpha\psi\|^2}\|Q_\beta P_\alpha\psi\|^2\,\|P_\alpha\psi\|^2$

$$= \|Q_\beta P_\alpha \psi\|^2 .$$

Obs. If $Q_\beta P_\alpha = P_\alpha Q_\beta$ $\forall \alpha,\beta$ then joint distr. indep. of order.
"A and B can be measured simultaneously".

Thm (extension of spectral thm to several). $A=A^\dagger$, $B=B^\dagger$
Iff $AB=BA$ then $\exists$ GONB $\{\phi_n\}$ of joint eigenvectors

__Sketch of proof__ If $AB = BA$, then B maps the eigenspace(α) to itself:

$$A\psi = \alpha\psi$$

$$\Rightarrow AB\psi = BA\psi = B\alpha\psi = \alpha B\psi.$$

Thus, in $\{\tilde{\phi}_n\}$ diag. A, B is block diag.

$$\begin{array}{c} \mathrm{eig}(\alpha_1) \\ \mathrm{eig}(\alpha_2) \\ \vdots \end{array} \begin{pmatrix} \ast & 0 & 0 \\ 0 & \ast & 0 \\ 0 & 0 & \ast \end{pmatrix}$$

$\square$

Thm Two observables $A=A^\dagger$, $B=B^\dagger$, with discrete spectrum; the joint distr $P_{A\,first}(\alpha,\beta)=P_{B\,first}(\alpha,\beta)$ iff $AB=BA$.

$A=\sum_\alpha \alpha P_\alpha$, $B=\sum_\beta \beta Q_\beta$ spectr. decomp.
sp. thm.

Pf "if": Suppose $AB=BA \overset{\text{sp. thm.}}{\Longrightarrow}$ diag. simul. $\Rightarrow P_\alpha Q_\beta = Q_\beta P_\alpha$.

$$P_{A\,first}(\alpha,\beta)=\|Q_\beta P_\alpha \psi\|^2,\quad P_{B\,first}(\alpha,\beta)=\|P_\alpha Q_\beta\psi\|^2.$$

"only if": Fix α,β. We suppose $\underline{\|PQ\psi\|^2=\|QP\psi\|^2}$ $\forall\psi\in\mathcal{H}$.

Want to show $\underline{\|(QP-PQ)\psi\|=0}$.

$$\psi = u + v, \quad \underline{Qu = u}, \quad \underline{Qv = 0}$$

$$\|(QP - PQ)(u+v)\|^2 = \langle u+v | (PQ - QP)(QP - PQ) | u+v\rangle$$

$$\boxed{(ST)^\dagger = T^\dagger S^\dagger}$$

$$= \langle u+v | PQP - QPQP - PQPQ + QPQ | u+v\rangle$$

$$= \langle u | \cancel{PQP} - \cancel{PQP} - \underline{PQP} + \underline{P} | u\rangle$$

$$\cancel{+ \langle u | PQP - PQP | v\rangle}$$

$$\cancel{+ \langle v | PQP - PQP | u\rangle}$$

$$+ \langle v | PQP | v\rangle$$

$$= \|Pu\|^2 - \|QPu\|^2 + \|QPv\|^2$$

$$= \|Pu\|^2 - \|PQu\|^2 + \cancel{\|PQv\|^2} = 0. \quad \square$$

$\underline{\text{Ex}}$ $[X_i, X_j] = 0$, $[A, B] = AB - BA$

$[P_i, P_j] = 0$

$[\sigma_2, \sigma_3] \neq 0$.

$[X_j, P_i] = i\hbar I \; \delta_{ij}$

Heisenberg's canonical commutation relation

"non-commuting A, B can't be measured simul.ly"

don't have sharp values simul.ly

Thm (Robertson–Schrödinger ineq.)

For any **bounded** self-adj A, B and any $\psi \in \mathcal{H}$ with $\|\psi\| = 1$,

$$\sigma_A \, \sigma_B \geq \frac{1}{2} \left| \langle \psi | \, [A, B] \, | \psi \rangle \right| \quad (*)$$

Rem unbounded: $A\psi$ exists for $\psi \in \text{domain}(A)$.

Thm: self-adj. A, B, $\psi \in \text{domain}(AB) \cap \text{domain}(BA)$ then $(*)$.

Exception: $A = X$, $B = P$. Even for **all** $\psi \in \mathcal{H}$

$$\sigma_X \, \sigma_P \geq \frac{\hbar}{2}.$$