Bell's thm

$$\phi \propto |n\text{-up}\rangle|n\text{-down}\rangle - |n\text{-down}\rangle|n\text{-up}\rangle$$

singlet state

Fact

$$p_{\underline{\alpha}\underline{\beta}} = \begin{bmatrix} \frac{1}{2}\sin^2(\theta/2) & \frac{1}{2}\cos^2(\theta/2) \\ \frac{1}{2}\cos^2(\theta/2) & \frac{1}{2}\sin^2(\theta/2) \end{bmatrix}$$

Pf $\phi = c\left(|\underline{\alpha}\text{-up}\rangle|\underline{\alpha}\text{-down}\rangle - |\underline{\alpha}\text{-down}\rangle|\underline{\alpha}\text{-up}\rangle\right)$

$|c| = \frac{1}{\sqrt{2}}$. Born's rule $\Rightarrow$ $\mathbb{P}(Z_1 = +1) = \frac{1}{2} = \mathbb{P}(Z_1 = -1)$.

If $Z_1 = +1$, $\phi \rightsquigarrow \phi'_+ = |\underline{\alpha}\text{-up}\rangle|\underline{\alpha}\text{-down}\rangle$

$Z_1 = -1$, $\phi \rightsquigarrow \phi'_- = |\underline{\alpha}\text{-down}\rangle|\underline{\alpha}\text{-up}\rangle$.

$$\mathbb{P}(Z_2 = +1 \mid Z_1 = +1) = |\langle \underline{\beta}\text{-up} \mid \underline{\alpha}\text{-down} \rangle|^2$$

$$= 1 - |\langle \underline{\beta}\text{-up} \mid \underline{\alpha}\text{-up} \rangle|^2$$

$$= 1 - \cos^2(\theta/2) = \sin^2 \frac{\theta}{2}$$

total prob $\mathbb{P}(Z_2 = +1, Z_1 = +1) = \mathbb{P}(\ \mid\ )\,\mathbb{P}(Z_1 = +1)$

$$= \frac{1}{2}\sin^2(\theta/2) = \frac{1}{2} - \frac{1}{2}\cos^2(\theta/2) = \frac{1}{2} - \frac{1}{4} - \frac{1}{4}\cos\theta$$

$$= \frac{1}{4} - \frac{1}{4}\underline{\alpha}\cdot\underline{\beta}. \qquad \square$$

$$\left[\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos^{\cancel{2}}(2x)\right]$$

$$\mu_{\underset{\smile}{\alpha},\underset{\smile}{\alpha}} = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} \qquad (\text{EPR})$$

<u>Pf of Bell's thm</u>: $\mu_{\underset{\smile}{\alpha}\underset{\smile}{\beta}}$ can't be created in a local way.

~~Ap~~ 1964 proof has 2 parts.

Part 1: EPR arg. $\exists$ hidden variables $Z_{1\underset{\smile}{\alpha}} = -Z_{2\underset{\smile}{\alpha}}$

Conclusion: $Z_{i\underset{\smile}{\alpha}} = \pm 1$, $Z_{1\underset{\smile}{\alpha}} = -Z_{2\underset{\smile}{\alpha}}$,

$\mathbb{P}(Z_{1\underset{\smile}{\alpha}} \neq Z_{2\underset{\smile}{\beta}}) = \cos^2(\theta/2)$

Part 2: $\underline{a}, \underline{b}, \underline{c}$ unit vectors in $\mathbb{R}^3$

$$\mathbb{P}\left(Z_{1\underline{a}} = Z_{1\underline{b}} \text{ or } Z_{1\underline{b}} = Z_{1\underline{c}} \text{ or } Z_{1\underline{c}} = Z_{1\underline{a}} \right) = 1$$

$$\Rightarrow \mathbb{P}(Z_{1\underline{a}} = Z_{1\underline{b}}) + \mathbb{P}(Z_{1\underline{b}} = Z_{1\underline{c}}) + \mathbb{P}(Z_{1\underline{c}} = Z_{1\underline{a}}) \geq 1$$

$$\Rightarrow \boxed{\mathbb{P}(Z_{1\underline{a}} \neq Z_{2\underline{b}}) + \mathbb{P}(Z_{1\underline{b}} \neq Z_{2\underline{c}}) + \mathbb{P}(Z_{1\underline{c}} \neq Z_{2\underline{a}}) \geq 1.}$$

Bell's inequality.

$$\cos^2(\theta_{\underline{ab}}/2) + \cos^2(\theta_{\underline{bc}}/2) + \cos^2(\theta_{\underline{ca}}/2)$$

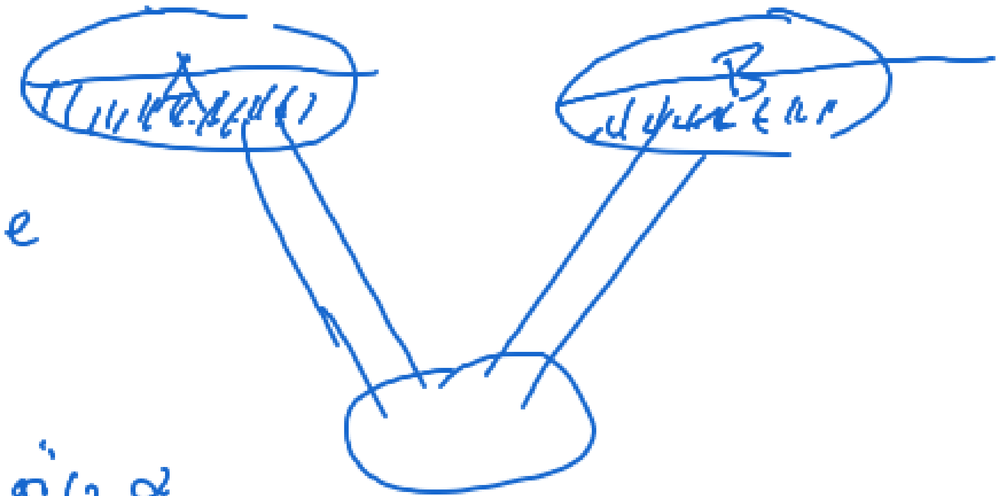
$\theta = 120^\circ$  $3\cos^2 60^\circ = \frac{3}{4}$ ↯

□

Bell's 1976 proof

λ = all variables in nature
about the particles,
not including Alice's choice $\underline{\alpha}$
or Bob's choice $\underline{\beta}$. $\rho(\lambda)$

$\mathbb{P}(Z_1 = z_1, Z_2 = z_2 \mid \underline{\alpha}, \underline{\beta}, \lambda)$



locality $\Rightarrow$ 1) $\sum_{z_2} \mathbb{P}(Z_1=z_1, Z_2=z_2 \mid \underline{\alpha}, \underline{\beta}, \lambda)$

is indep of $\underline{\beta}$ $=: f_1(z_1, \underline{\alpha}, \lambda)$

and 2) $\mathbb{P}(Z_2=z_2 \mid Z_1=z_1, \underline{\alpha}, \underline{\beta}, \lambda)$

is indep. of z_1 or $\underline{\alpha}$ $= f_2(z_2, \underline{\beta}, \lambda)$

$$\Rightarrow \boxed{\mathbb{P}(Z_1=z_1, Z_2=z_2 \mid \underline{\alpha}, \underline{\beta}, \lambda) = f_1(z_1, \underline{\alpha}, \lambda)\, f_2(z_2, \underline{\beta}, \lambda)}$$

loc. condition

obs. probs: $\mathbb{P}(Z_1 = z_1, Z_2 = z_2 \mid \underline{\alpha}, \underline{\beta})$

$$= \int d\lambda\, \rho(\lambda)\, \mathbb{P}(Z_1 = z_1, Z_2 = z_2 \mid \underline{\alpha}, \underline{\beta}, \lambda)$$

Correlation coeff. $\kappa(X, Y) = \dfrac{\mathbb{E}[(X - \mathbb{E}X)(Y - \mathbb{E}Y)]}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}}$

$\mathrm{Var}(X) = \mathbb{E}[(X - \mathbb{E}X)^2]$

obs. corr. coeff.

$$K = \sum_{z_1, z_2 = \pm 1} z_1 z_2\, \mathbb{P}(Z_1 = z_1, Z_2 = z_2)$$

$$K = K(\underline{\alpha}, \underline{\beta})$$

$\underline{\text{Prop}}$ loc $\Rightarrow$ $\left|\kappa(\underline{\alpha},\underline{\beta})+\kappa(\underline{\alpha},\underline{\beta}')+\kappa(\underline{\alpha}',\underline{\beta})-\kappa(\underline{\alpha}',\underline{\beta}')\right|\leq 2$

(Bell's ineq., CHSH ineq.)

$\underline{\text{Pf}}$ $\kappa(\underline{\alpha},\underline{\beta})=\int d\lambda\, \rho(\lambda)\sum_{z_1}\sum_{z_2} z_1 z_2\, \mathbb{P}(Z_1=z_1, Z_2=z_2 \mid \underline{\alpha},\underline{\beta},\lambda)$

$=\int d\lambda\, \rho(\lambda)\sum_{z_1 z_2} z_1 z_2\, f_1(z_1,\underline{\alpha},\lambda)\, f_2(z_2,\underline{\beta},\lambda)$

$=\int d\lambda\, \rho(\lambda)\ \mathbb{E}(Z_1|\underline{\alpha},\lambda)\ \mathbb{E}(Z_2|\underline{\beta},\lambda)$.

$\left|\mathbb{E}(Z_i|\underset{\underline{\beta}}{\underline{\alpha}},\lambda)\right|\leq 1$

so

$$\left|K(\alpha,\beta)\pm K(\alpha,\beta')\right| = \left|\int d\lambda\,\rho(\lambda)\,\mathbb{E}(Z_1|\alpha,\lambda)\left(\mathbb{E}(Z_2|\beta\lambda)\pm\mathbb{E}(Z_2|\beta'\lambda)\right)\right|$$

$$\le \int d\lambda\,\rho(\lambda)\,\left|\mathbb{E}(Z_1|\alpha\lambda)\right|\,\left|\mathbb{E}(Z_2|\beta\lambda)\pm\mathbb{E}(Z_2|\beta'\lambda)\right|$$

For $u,v\in[-1,1]$, $\quad |u+v|+|u-v|\le 2$ $\leftarrow$

$$-u-v\quad -u+v = -2u \le 2$$

Hence, $\left|K(\alpha,\beta)+K(\alpha,\beta')+K(\alpha'\beta)-K(\alpha'\beta')\right|$

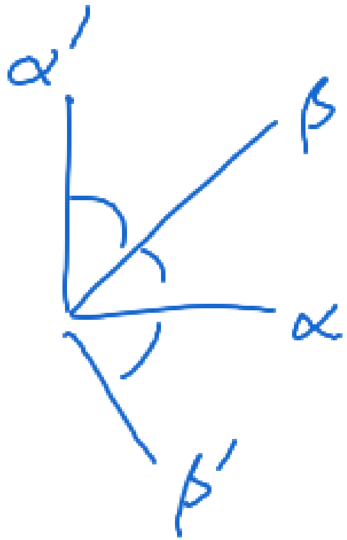
$$\le \left|\quad + \quad{}'\right| + \left|\quad{}'\quad - \quad{}''\quad\right|$$

$$\le \underbrace{\int d\lambda\,\rho(\lambda)}_{\le 2}\left(\left|\mathbb{E}(Z_2|\beta\lambda)+\mathbb{E}(Z_2|\beta'\lambda)\right| + \left|\mathbb{E}(Z_2|\beta\lambda)-\mathbb{E}(Z_2|\beta'\lambda)\right|\right)$$

$\square$

Now

$K(\alpha,\beta) = -\alpha\cdot\beta = -\cos\theta$



$$\Longrightarrow \quad K(\alpha,\beta) + K(\alpha,\beta') + K(\alpha',\beta) - K(\alpha'\beta') = -2\sqrt{2}.$$