POVMs and PVMs

A self-adjoint corresponds PVM E on $\mathbb{R}$.

<u>Thm</u> (spectral thm) $\forall$ s-a. A $\exists_1$ PVM E on $\mathbb{R}$ ("spectral PVM" = "spectral measure") such that

$$A = \int_{\mathbb{R}} \alpha \, E(d\alpha).$$

<u>Rem</u> analog to $A = \sum_\alpha \alpha P_\alpha$

$$\int_{\mathbb{R}} f(x)\,\mu(dx), \quad \int_{\mathbb{R}} f(x)\,E(dx), \quad E(B) = 1_B(A)$$

Thm:
Commuting s.-a. $A_1, \dots, A_n \implies \exists_1$ PVM E on $\mathbb{R}^n$ such that

$$A_k = \int_{\mathbb{R}^n} \alpha_k \, E(d\underline{\alpha})$$

<u>Ex</u> 3 pos. op.s X_1, X_2, X_3 on $L^2(\mathbb{R}^3, \mathbb{C})$

$$E(B) = 1_B, \quad B \subseteq \mathbb{R}^3$$

<u>Ex</u> consecutive quantum measurements $A_1, \dots, A_n$

$0 < t_1 < t_2 < \dots < t_n$ $\quad \mathbb{P}\Big((Z_1, \dots Z_n) \in B \Big) = \langle \psi_0 | E(B) | \psi_0 \rangle$

$B \subseteq \mathbb{R}^n$ POVM

$$E(B) = \sum_{(z_1 \ldots z_n) \in B \cap S} R^\dagger R \,, \quad R = P_{n z_n} \cdots P_{2 z_2} U_{t_2 - t_1} P_{1 z_1} U_{t_1 - t_0}$$

$$A_j = \sum_{z_j} z_j P_{j z_j}$$

$$S = \text{spectrum}(A_1) \times \ldots \times \text{spectrum}(A_n)$$

<u>Ex</u> GRWf, $F = \{ ((t_1, \underline{x}_1, i_1), (t_2, \underline{x}_2, i_2), \ldots) \in Z$

$$Z = \{ F \in (\mathbb{R}^4 \times \{1 \ldots N\})^\infty : 0 < t_1 < t_2 < \ldots \}$$ history space

$$\mathbb{P}(F \in B) = \langle \Psi_0 | G(B) | \Psi_0 \rangle$$

POVM

• 2 flashes, 1 particle

$$G(dt_1 \times d^3\underline{x}_1 \times dt_2 \times d^3\underline{x}_2) = 1_{0<t_1<t_2}\, e^{-\lambda t_2}\, \lambda^2 \times$$

$$e^{iHt_1} C(\underline{x}_1) e^{iH(t_2-t_1)} C(\underline{x}_2) \quad C(\underline{x}_2) e^{-iH(t_2-t_1)} C(\underline{x}_1) e^{-iHt_1}$$

$$dt_1\, d^3\underline{x}_1\, dt_2\, d^3\underline{x}_2 \,.$$

Main thm about POVMs For every quantum experiment $\mathcal{E}$ on a system S with outcomes in Z ~~such~~ $\exists_1$ POVM E : if S has ψ, $\mathbb{P}(Z \in B) = \langle \psi | E(B) | \psi \rangle$.

$B \subseteq Z$

$\mathcal{E}$ begins at t_i, ends at t_f

$$\Psi_{S\cup A}(t_i) = \psi(t_i) \otimes \phi_A(t_i)$$

<u>BM</u>: $Z = \zeta(Q(t_f))$ ζ = calibration fct

"Law of operators": For $\mathcal{E}$ def'd by $\phi_A, t_i, t_f, H_{S\cup A}$,

$$E_Z \psi = \langle \phi_A | U^+ P_{B_Z} U(\psi \otimes \phi_A) \rangle_y$$

$$U = e^{-iH_{S\cup A}(t_f - t_i)}$$

$$B_Z = \{ q \in \mathbb{R}^{3N} : \zeta(q) = z \}$$

partial inner prod: $\Psi(x,y)$, $\phi(y)$

$$\langle \phi | \Psi \rangle_y = \int dy \; \phi^*(y) \, \Psi(x,y) = f(x)$$

Pf of the main thm

$$\mathbb{P}(Z=z) = \mathbb{P}\left(Q(t_f) \in B_z\right)$$

$$= \int_{\cancel{\mathbb{R}^{3N}}\, B_z} dq \, |\Psi(q, t_f)|^2$$

$$= \langle \underline{\Psi}(t_f) | \, P_{B_z} \, | \underline{\Psi}(t_f) \rangle$$

$$= \langle \psi \otimes \phi_A | U^\dagger P_{B_z} U (\psi \otimes \phi_A) \rangle_{S \cup A}$$

$$= \langle \psi | E_z | \psi \rangle_S . \qquad \square$$

Pf from GRWf

$\mathbb{P}(F \in B) = \langle \Psi(t_i) | G(B) | \Psi(t_i) \rangle$

$Z = \zeta(F)$, $B_z = \{ f : \zeta(f) = z \}$. Then

$$\mathbb{P}(Z = z) = \mathbb{P}(F \in B_z)$$

$$= \langle \underbrace{\Psi(t_i)}_{\psi \otimes \phi_A} | G(B_z) | \underbrace{\Psi(t_i)}_{\psi \otimes \phi_A} \rangle$$

$$= \langle \psi | E_z^{GRW} | \psi \rangle \text{ with}$$

$$E_z^{GRW} \psi = \langle \phi_A | G(B_z) | \psi \otimes \phi_A \rangle_y \neq E_z \quad \square$$

Limitations to knowledge

Corollary $\nexists$ experiment with $\cancel{Z} = \psi$ or $Z = \mathbb{C}\psi$.

Pf Suppose otherwise, $Z = \psi$. Then, for any ψ,
Z is deterministic $\mathbb{P}_\psi(Z \in B) = 1_{\psi \in B}$,
$\rho_Z(\phi) = \delta(\phi - \psi) \neq \langle \psi | E_\phi | \psi \rangle$ for any POVM. $\square$

<u>Corollary</u> $\nexists$ experiment in BM that can measure the instantaneous velocity of a particle with unknown wave fct.

<u>Pf</u> $Q \sim |\psi|^2$, $v = \Im \frac{\nabla \psi}{\psi}(Q)$

distr. of v is not quadratic in ψ $\square$

<u>Ex</u> asymptotic velocity $\sim |\hat{\psi}(k)|^2$

$$k \leftarrow \frac{u}{m\hbar}$$