Absorbing Boundary Rule

$$i\hbar \frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi \quad \text{in } \Omega$$

$$\frac{\partial\psi}{\partial n} = i\kappa\psi \quad \text{on } \partial\Omega$$

(cf. Neumann b.c. $\frac{\partial\psi}{\partial n} = 0$)

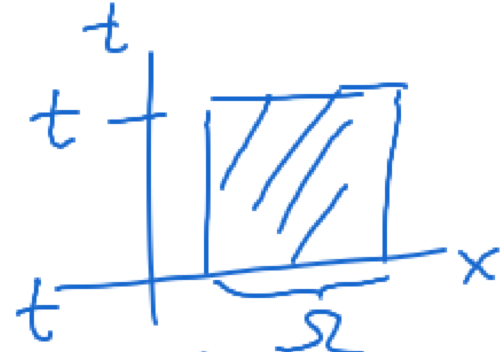
$$\mathbb{P}(t_1 < T < t_2,\ \underline{X} \in B) = \int_{t_1}^{t_2} dt \int_B d^2\underline{x}\ \underline{n}(\underline{x}) \cdot \underline{j}(\underline{x}, t)$$

$$\mathbb{P}(Z = \infty) = 1 - \int_0^\infty dt \int_{\partial\Omega} d^2\underline{x}\ \underline{n} \cdot \underline{j} .$$

prob. conserv. (Ostrogradski–Gauss int. thm.)

4-vector $j = (\rho, \underline{j})$

$$\mathrm{div}_4 j = \frac{\partial \rho}{\partial t} + \nabla \cdot \underline{j} = 0 \quad \text{in } \Omega$$



$$0 = \int_0^t dt' \int_\Omega d^3\underline{x}\; \mathrm{div}_4\, j(t', \underline{x})$$

$$= \int_\Omega d^3\underline{x}\; \rho(\underline{x}, t) - \int_\Omega d^3\underline{x}\; \rho(\underline{x}, 0) + \underbrace{\int_0^t dt' \int_{\partial\Omega} d^2\underline{x}\; \underline{n}\cdot\underline{j}(t', \underline{x})}_{\geq 0}$$

$$= \|\psi_t\|^2 - 1 +$$

$\cancel{1 - \|\psi_t\|^2 \geq 0}$ $\quad \|\psi_t\|^2 \leq 1, \quad t \mapsto \|\psi_t\|$ decreasing and $\lim_{t\to\infty} \|\psi_t\|^2 = 1 - \underbrace{\int_0^{\cancel{t}\,\infty} dt' \int_{\partial\Omega} d^2\underline{x}\; \underline{n}\cdot\underline{j}}_{\geq 0}$

<u>POVM</u>

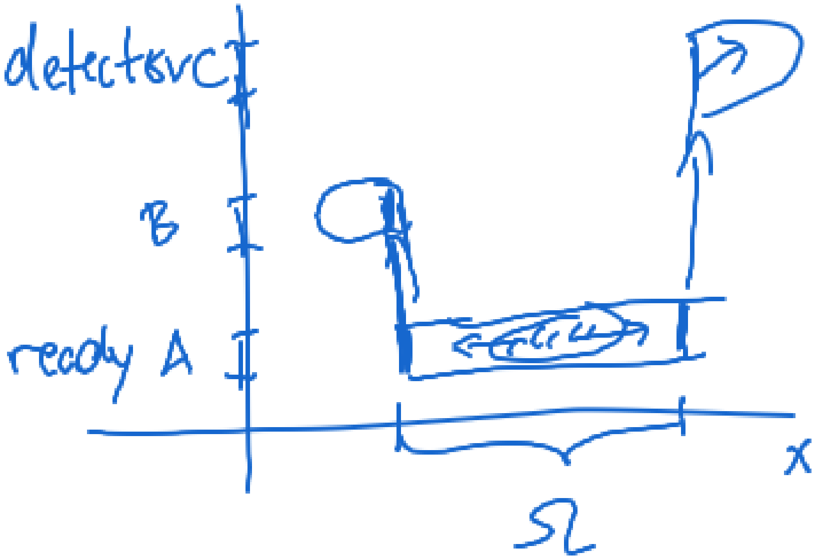
$W_t \psi_0 = \psi_t$ contraction semigroup

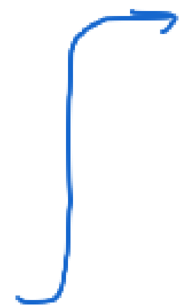
$$E(dt \times d^2\underline{x}) = \frac{\hbar\kappa}{m} W_t^\dagger |\underline{x}\rangle\langle\underline{x}| W_t \;\, dt\, d^2\underline{x}$$

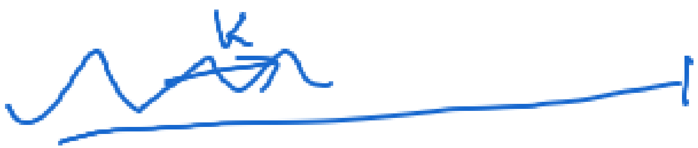
$$E(\{\infty\}) = \lim_{t\to\infty} W_t^\dagger W_t$$

$$Z = [0,\infty)\times\partial\Omega \cup \{\infty\}$$

Derivation or motivation







$$\Omega = (-\infty, 0]$$

$$\psi(x) = \left(e^{ikx} + c_k e^{-ikx}\right) e^{-i\omega t}$$

b.c. $\psi'(0) = i\kappa \psi(0)$

$|c_k|^2$ = strength of reflected wave

$A_k = 1 - |c_k|^2$ = absorbed

Time-energy-uncertainty relation

$\sigma_T \sigma_E \geq \frac{\hbar}{2}$. Fix Ω, T ~~d~~ abs. bdy rule

Suppose $\mathbb{P}(Z=\infty)=0$. prob dist. on $[0,\infty)$, σ_T = std. dev.

$W_t = e^{-iHt}$, H not self-adj., not normal, no spectral PVM

$\psi \in L^2(\Omega)$, $\psi \in L^2(\mathbb{R}^3)$, $H_{free} = -\frac{\hbar^2}{2m}\nabla^2 + V$ on $\mathbb{R}^3$

$\longrightarrow$ prob. distr. on $\mathbb{R}$, σ_E = std. dev.

__Thm__ Suppose $\Omega \subset \mathbb{R}^3$ is open, $\partial\Omega$ is sufficiently regular and such that $\lim_{t\to\infty} W_t^+ W_t = 0$

Then $\forall \psi \in C^\infty_{comp}(\Omega)$, $\sigma_T \sigma_E \geq \frac{\hbar}{2}$.

What if $\mathbb{P}(Z=\infty) > 0$? Then $\sigma_T = \infty$.

$$\tilde{\sigma}_T := \sqrt{Var(T \mid Z \neq \infty)}$$

__Thm__ If $\Omega \subset \mathbb{R}^3$ open, $\partial\Omega$ regular, $\psi \in C^\infty_{comp}(\Omega)$

Then $\tilde{\sigma}_T \sigma_E \geq \sqrt{\mathbb{P}(Z \neq \infty)}\, \frac{\hbar}{2}$

History

Aharonov & Bohm 1961.

$$x + \frac{tp}{m} = 0 \Rightarrow t = -\frac{xm}{p}$$

$$\Omega = (-\infty, 0]$$

$$x<0, \; p>0$$

$$\hat{T} = -\hat{P}^{-1/2}\hat{X}\, m\, \hat{P}^{-1/2}$$

Werner 1987 : b.c. $\frac{\partial\psi}{\partial n} = i\kappa\psi$, $\mathbb{P}(T)$

Density Matrix and Mixed State

$$\mathbb{S}(\mathcal{H}) = \{\psi \in \mathcal{H} : \|\psi\| = 1\}$$

random wf $\underline{\Psi}$ in $\mathbb{S}(\mathcal{H})$, prob. dist. μ on $\mathbb{S}(\mathcal{H})$

<u>Ex</u> $\mu = u$ for $\dim \mathcal{H} < \infty$.

<u>Claim</u> It is impossible to determine μ empirically.

Sometimes $\mu_1 \neq \mu_2$ are empirically indistinguishable.

$$\underline{\mathbb{P}(Z \in B)} = \mathbb{E}_{\Psi} \langle \Psi | E(B) | \Psi \rangle = \int_{\mathbb{S}(\mathcal{H})} \mu(d\psi) \langle \psi | E(B) | \psi \rangle$$

<u>trace</u>

$$\operatorname{tr} T = \sum_n \langle n | T | n \rangle$$

$|n\rangle$ ONB.