Bohr's Reply to EPR

B vs. B' ambiguity

mistake in argument

$$\frac{\begin{array}{c} A \Rightarrow B \\ B' \Rightarrow C \end{array}}{A \Rightarrow C}$$

doubt a premise $\quad B' \Rightarrow C$

Trace

$|n\rangle$ ONB

$$\operatorname{tr} T = \sum_n \langle n|T|n\rangle$$

$T \geq 0$: $\operatorname{tr} T \in \cancel{\mathbb{R}}$, $[0,\infty)$ or $\operatorname{tr} T = \infty$

Trace class $= \{T : \mathcal{H} \to \mathcal{H} \mid \operatorname{tr}\sqrt{T^+T} < \infty\}$

$\Rightarrow \operatorname{tr} T$ indep. of ONB.

Properties (i) linear $\operatorname{tr}(T+S) = \operatorname{tr} T + \operatorname{tr} S$

$\operatorname{tr}(\lambda T) = \lambda \operatorname{tr} T$, $\lambda \in \mathbb{C}$

(ii) $\mathrm{tr}(AB\cdots YZ) = \mathrm{tr}(ZAB\cdots Y)$

in particular, $\mathrm{tr}(AB) = \mathrm{tr}(BA)$

but $\mathrm{tr}(ABC) \neq \mathrm{tr}(CBA)$ in general

$= \mathrm{tr}(CAB) = \mathrm{tr}(BCA)$

(iii) If T diagonalized, $\mathrm{tr}\, T = \Sigma$ eigenvalues

(iv) $\mathrm{tr}(T^\dagger) = (\mathrm{tr}\, T)^*$

(v) $T = T^\dagger \Rightarrow \mathrm{tr}\, T \in \mathbb{R}$

(vi) $T \geq 0 \Rightarrow \mathrm{tr}\, T \geq 0$.

(vii) $\|\psi\| = 1$, $\mathrm{tr}(|\psi\rangle\langle\psi| E) = \langle\psi|E|\psi\rangle$.

Trace Formula

μ on $\mathbb{S}(\mathcal{H})$, $\mathcal{E}$ with $E(\cdot)$, Z

$$\mathbb{P}(Z \in B) = \mathbb{E}\langle \Psi | E(B) | \Psi \rangle$$

$$= \int_{\mathbb{S}(\mathcal{H})} \mu(d\psi) \underbrace{\langle \psi | E(B) | \psi \rangle}_{tr\left(|\psi\rangle\langle\psi| E(B) \right)}$$

$$= tr\left(\Big[\underbrace{\int_{\mathbb{S}(\mathcal{H})} \mu(d\psi)\, |\psi\rangle\langle\psi|}_{\rho_\mu} \Big] E(B) \right)$$

$$= tr\left(\rho_\mu E(B) \right).$$

ρ_μ = density matrix of μ.

- if $\mu_1 \neq \mu_2$ have $\rho_{\mu_1} = \rho_{\mu_2}$, they are empirically indistinguishable.

Ex $\mu = u$ uniform on $\mathbb{S}(\mathcal{H})$, $\dim \mathcal{H} = d < \infty$.

$$\rho_u = \int u(d\psi) |\psi\rangle\langle\psi| = \frac{1}{d} I$$

b/c $\quad U \rho_u U^{-1} = \rho_u \quad \forall U \in \mathcal{U}(\mathcal{H})$.

$$\operatorname{tr} \rho = \operatorname{tr} \int \mu(d\psi) |\psi\rangle\langle\psi| = \int \mu(d\psi) \operatorname{tr} |\psi\rangle\langle\psi| = \int \mu(d\psi) 1 = 1.$$

Ex μ : ONB $\phi_1 \cdots \phi_d$

$$\mu(\{\phi_j\}) = \frac{1}{d}.$$

$$\rho_\mu = \int \mu(d\psi)\, |\psi\rangle\langle\psi| = \sum_{j=1}^{d} \frac{1}{d} |\phi_j\rangle\langle\phi_j| = \frac{1}{d} I$$

Prop ρ $\quad \exists \mu : \rho = \rho_\mu \iff \begin{cases} \operatorname{tr} \rho = 1 \\ \rho \geq 0 \end{cases}$

Pf $\operatorname{tr} \rho_\mu = 1.$

$$\langle \phi | \rho_\mu | \phi \rangle = \int \mu(d\psi)\, \underbrace{\langle\phi|\psi\rangle\langle\psi|\phi\rangle}_{\geq 0} \geq 0.$$

$$\rho = \sum_j p_j \quad |\phi_j\rangle\langle\phi_j| \quad , \qquad p_j \geq 0$$

$$\sum p_j = \operatorname{tr} \rho = 1$$

$$\mu = \sum_j p_j \, \delta_{\phi_j}$$

$$\rho_\mu = \int \mu(d\psi) \, |\psi\rangle\langle\psi| = \sum_j p_j \, |\phi_j\rangle\langle\phi_j| = \rho .$$

$\square$

Dynamics

$$\Psi_0 \sim \mu, \quad \Psi_t = \overbrace{e^{-iHt/\hbar}}^{U_t} \Psi_0, \quad \mu_0 \rightsquigarrow \mu_t$$

$$\Psi_t \sim \mu_t$$

$$\rho_t := \rho_{\mu_t} = \int \mu_t(d\psi)\, |\psi\rangle\langle\psi| = \int \mu_0(d\psi) \quad U_t|\psi\rangle \underbrace{\langle\psi|U_t^\dagger}_{\langle U_t\psi|}$$

$$= \underline{U_t \big(\overbrace{\rho_{\mu_0}}^{=:\rho_0} \big) U_t^\dagger}, \quad \frac{d}{dt} e^{-iHt/\hbar} = -\frac{i}{\hbar} H e^{-iHt/\hbar}$$

$$= e^{-iHt/\hbar}\left(-\frac{i}{\hbar}H\right)$$

$$\frac{d\rho_t}{dt} = -\frac{i}{\hbar} H U_t \rho_0 U_t^\dagger + U_t \rho_0 U_t^\dagger \frac{i}{\hbar} H$$

$$= -\frac{i}{\hbar} H \rho_t + \frac{i}{\hbar} \rho_t H$$

$$= -\frac{i}{\hbar} \left[H, \rho_t \right]$$ von Neumann eq. (1927)

<u>Def</u> If $\rho = |\psi\rangle\langle\psi|$ (1d proj.), then ρ is called a "pure state",

$\rho = \rho_\mu$ with μ concentrated on $\mathbb{C}\psi$.

Otherwise, ρ is called a "mixed state".

Reduced Density Matrix

bipartite $a \cup b$, $\mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_b$

$\psi_a \otimes \psi_b \neq \phi \in \mathcal{H}$

$T : \mathcal{H}_a \to \mathcal{H}_a$, $\langle n_a | T | m_a \rangle = T_{n_a m_a}$

Now $T : \mathcal{H} \to \mathcal{H}$, $|n_a\rangle$ ONB of $\mathcal{H}_a$, $|n_b\rangle$ ONB of $\mathcal{H}_b$,

$|n_a\rangle \otimes |n_b\rangle$ ONB of $\mathcal{H}_a \otimes \mathcal{H}_b$.

$$\langle \psi \otimes \phi | \chi \otimes \omega \rangle_{\mathcal{H}} = \langle \psi | \chi \rangle_a \langle \phi | \omega \rangle_b$$

$\psi \in \mathcal{H}$, $\psi = \sum_{n_a, n_b} \psi_{n_a n_b} |n_a\rangle |n_b\rangle$

$$\langle \psi | \phi \rangle = \sum_{n_a n_b} \psi^*_{n_a n_b} \phi_{n_a n_b}$$

$$= \int_{\Omega_a} d\omega_a \int_{\Omega_b} d\omega_b \psi^*(\omega_a, \omega_b) \phi(\omega_a, \omega_b)$$

$\mathcal{H}_a = L^2(\Omega_a)$

$\mathcal{H}_b = L^2(\Omega_b)$

$\mathcal{H} = L^2(\Omega_a \times \Omega_b)$

$T : \mathcal{H} \to \mathcal{H}$, $\quad T_{n_a n_b m_a m_b}$

$$T = T_a \otimes T_b \iff T_{n_a n_b m_a m_b} = T_{a\, n_a m_a} \, T_{b\, n_b m_b}$$

2 non-interacting systems: $H = H_a \otimes I_b + I_a \otimes H_b$

$$\Rightarrow \quad U_t = e^{-iHt/\hbar}, \quad U_t = \underbrace{U_{at}}_{e^{-iH_a t/\hbar}} \otimes \underbrace{U_{bt}}_{e^{-iH_b t/\hbar}}$$

<u>Claim</u>: (Main thm) Suppose apparatus inter-acts only on a, not on b. Then $E(B) = E_a(B) \otimes I_b$.