reduced density matrix

bipartite

system a | system b

app

no int.

$$\mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_b$$

Main thm about POVMs: $E(B) = E_a(B) \otimes I_b$

$$\mathbb{P}(Z \in B) = \langle \psi | E(B) | \psi \rangle = \mathrm{tr}\left(\rho_\psi E_a(B)\right)$$

Def reduced density matrix

$$\rho_\psi = \mathrm{tr}_b \, |\psi\rangle\langle\psi| \quad \text{partial trace}$$

Lev Landau (1927)

Partial trace

$\{\phi^a_{n_a}\}_{n_a}$ ONB of $\mathcal{H}_a$, $\{\phi^b_{n_b}\}_{n_b}$ ONB of $\mathcal{H}_b$ $\Rightarrow$ $\{\phi^a_{n_a}\otimes\phi^b_{n_b}\}_{n_a,n_b}$ ONB of $\mathcal{H}$

$T:\mathcal{H}\to\mathcal{H}$, $T_{n_a n_b m_a m_b}$

Def $S=\mathrm{tr}_b T \iff S_{n_a m_a}=\sum_{n_b} T_{n_a n_b m_a m_b}\Big|_{m_b=n_b}$

$[S:\mathcal{H}_a\to\mathcal{H}_a]$

$$\iff \langle\phi^a_{n_a}|S|\phi^a_{m_a}\rangle=\sum_{n_b}\langle\phi^a_{n_a}\otimes\phi^b_{n_b}|T|\phi^a_{m_a}\otimes\phi^b_{n_b}\rangle$$

$$\iff S=\sum_{n_b}\langle\phi^b_{n_b}|T|\phi^b_{n_b}\rangle_{\mathcal{H}_b}$$

partial inner prod.

Properties

1) linear $\mathrm{tr}_b(R+T) = \mathrm{tr}_b R + \mathrm{tr}_b T$

$\lambda \in \mathbb{C}$: $\mathrm{tr}_b(\lambda T) = \lambda\, \mathrm{tr}_b(T)$

2) $\mathrm{tr}(\mathrm{tr}_b T) = \mathrm{tr}\, T$, $\quad \mathrm{tr}_b : \mathrm{TRCL}(\mathcal{H}) \longrightarrow \mathrm{TRCL}(\mathcal{H}_a)$

3) $\mathrm{tr}_b(T^\dagger) = (\mathrm{tr}_b T)^\dagger$

4) $\mathrm{tr}_b(T_a \otimes T_b) = T_a\,(\mathrm{tr}\, T_b)$

5) If $T \geq 0$ then $\mathrm{tr}_b T \geq 0$

$$\left(\langle \psi_a | (\mathrm{tr}_b T) | \psi_a \rangle = \sum_{n_b} \langle \psi_a \otimes \phi^b_{n_b} | T | \psi_a \otimes \phi^b_{n_b} \rangle \geq 0 \right)$$

$$6)\quad \operatorname{tr}_b\left[R\left(T_a \otimes I_b\right)\right] = \left(\operatorname{tr}_b R\right) T_a$$

$$7)\quad \operatorname{tr}_b\left[R\left(I_a \otimes T_b\right)\right] = \operatorname{tr}_b\left[\left(I_a \otimes T_b\right) R\right]$$

$$8)\quad \operatorname{tr}\left[R\left(T_a \otimes I_b\right)\right] = \operatorname{tr}\left[\left(\operatorname{tr}_b R\right) T_a\right]$$

$$\Rightarrow\quad R = |\psi\rangle\langle\psi| \,,\quad \psi \in \mathcal{H}_a \otimes \mathcal{H}_b \,,\quad T_a = E_a(B)$$

$$\mathbb{P}(Z \in B) = \langle\psi|\, E(B)\, |\psi\rangle = \langle\psi|\, E_a(B) \otimes I_b \,|\psi\rangle$$

$$= \operatorname{tr}_{\substack{\cancel{\text{sub}} \\ \mathcal{H}}}\left(|\psi\rangle\langle\psi| \; E_a(B) \otimes I_b \right) \overset{8)}{=} \operatorname{tr}_{\mathcal{H}_a}\left(\left(\operatorname{tr}_b |\psi\rangle\langle\psi|\right) E_a(B)\right)$$

$$\square$$

Rem $\rho_\psi \geq 0$, $\operatorname{tr} \rho_\psi = 1$

Pf: $|\psi\rangle\langle\psi| \geq 0$, $\operatorname{tr}(\operatorname{tr}_b R) = \operatorname{tr} R$

$= \operatorname{tr} |\psi\rangle\langle\psi| = 1$. $\square$

Rem If $\rho \geq 0$, $\operatorname{tr} \rho = 1$, then $\exists \psi \in S(\mathcal{H}_a \otimes \mathcal{H}_b)$:

$\rho : \mathcal{H}_a \to \mathcal{H}_a$ $\qquad \rho_\psi = \rho$

(provided $\dim \mathcal{H}_b$ large enough).

Pf: $\rho = \sum_n p_n |\phi_n\rangle\langle\phi_n|$, $p_n \geq 0$, $\sum_n p_n = 1$, ONB ϕ_n of $\mathcal{H}_a$

Choose ONB $\{\chi_n\}_n$ of $\mathcal{H}_b$, set $\psi = \sum_n \sqrt{p_n}\, \phi_n \otimes \chi_n$

Then $\|\psi\|^2 = \sum_{nm} |c_{nm}|^2 = \sum_n |\sqrt{p_n}|^2 = \sum_n p_n = 1.$

$$\mathrm{tr}_b |\psi\rangle\langle\psi| = \sum_m \underbrace{\langle\chi_m|\psi\rangle}_{\sqrt{p_m}|\phi_m\rangle}\underbrace{\langle\psi|\chi_m\rangle}_{\langle\phi_m|\sqrt{p_m}} = \sum_m p_m |\phi_m\rangle\langle\phi_m| = \rho.$$

$\square$

Statistical reduced density matrix

$\Psi \in \mathcal{H}_a \otimes \mathcal{H}_b$ random $\sim \mu$,

$$\rho = \mathbb{E}_\mu \mathrm{tr}_b |\Psi\rangle\langle\Psi| = \mathrm{tr}_b \mathbb{E}_\mu |\Psi\rangle\langle\Psi|$$

Measurement problems and density matrix

$$\Psi = \sum_\alpha \Psi_\alpha \,, \quad \Psi_\alpha = c_\alpha \psi_\alpha \otimes \phi_\alpha$$

$$c_\alpha = \| P_\alpha \psi \|, \quad \psi_\alpha = \frac{P_\alpha \psi}{\| P_\alpha \psi \|} \,,$$

ϕ_α disj. support

$\phi_{\alpha_1} \perp \phi_{\alpha_2} \quad \forall \alpha_1 \neq \alpha_2$

ONB $\{\phi_n\}_n$ of $\mathcal{H}_{app}$

red d.m. of obj

$$\rho_\Psi = \operatorname{tr}_{app} |\Psi\rangle\langle\Psi| = \sum_n \langle \phi_n | \Psi \rangle \langle \Psi | \phi_n \rangle$$

$$= \sum_\alpha |c_\alpha|^2 \, |\psi_\alpha\rangle\langle\psi_\alpha|$$

<u>Want</u> ψ^{obj}_{after} random $\mu = \begin{cases} \psi_\alpha & \text{w/ prob. } |c_\alpha|^2 \\ \end{cases}$

$$\mu = \sum_\alpha |c_\alpha|^2 \delta_{\psi_\alpha}, \quad \rho_\mu = \sum_\alpha |c_\alpha|^2 |\psi_\alpha\rangle\langle\psi_\alpha|$$

<u>So</u> $\rho_{\underline{\Psi}} = \rho_\mu$ conclude: no problem

<u>Argument is incorrect</u> equal d.m. does not mean equal physical situation.

Def Decoherence $\iff$ $\rho_\psi = \sum p_\alpha |\phi_\alpha\rangle\langle\phi_\alpha|$

Def also for red. d. m.

ρ_ψ is pure iff $\rho_\psi = |\phi\rangle\langle\phi|$

mixed otherwise

Fact: ρ_ψ is pure iff $\psi = \psi_a \otimes \psi_b$

Quantum logic

1) math

2) analogy

3) philosophical idea

Logic = what is true in all situations.

more restricted: propositional logic

(rules for "and", "or", "not",

"$\forall x \in M$": $A(x) = \bigwedge_{x \in M} A(x)$

$\exists x \in M : A(x) = \bigvee_{x \in M} A(x)$.)

$A \wedge B = A$ and B (conjunction)

$A \vee B = A$ or B (disjunction)

$\neg A$ = not A

Def <u>Boolean algebra</u> set $\mathcal{A} = \{A, B, C, \ldots\}$

$\wedge, \vee, \neg,$ $A \wedge B,\ A \vee B,\ \neg A$

$\wedge : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$. such that

- $\wedge, \vee$ are associative, commutative, idempotent $A \wedge A = A,\ A \vee A = A$.
- absorption laws: $A \wedge (A \vee B) = A$
  $A \vee (A \wedge B) = A$
- $\exists\, 0 \in \mathcal{A}, 1 \in \mathcal{A} : \forall A \in \mathcal{A} :$ $A \wedge 0 = 0,\ A \wedge 1 = A$
  $A \vee 0 = A,\ A \vee 1 = 1$.
- complementation: $A \wedge \neg A = 0,\ A \vee \neg A = 1$
- distributive: $A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C),\ A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C)$.

Conseq $\neg(\neg A) = A$

de Morgan's laws $(\neg A) \vee (\neg B) = \neg(A \wedge B)$

$$(\neg A) \wedge (\neg B) = \neg(A \vee B)$$

- A, B, C propositions $\Rightarrow$ Boo. alg.
- A, B, C subsets of Ω $\Longrightarrow$ Boo. alg.

  $A \wedge B = A \cap B$, $A \vee B = A \cup B$, $\neg A = A^c = \Omega \setminus A$
- A, B, C closed subspaces of $\mathcal{H}$ $\Longrightarrow$ almost Boo. alg.

  $A \wedge B = A \cap B$, $A \vee B = \overline{\text{span}}(A \cup B)$, $\neg A = A^\perp$

  "joint" "orthogonal complement"

  $= \{ \phi \in \mathcal{H} \mid \forall \psi \in A : \langle \psi | \phi \rangle = 0 \}$

lattice = orthomodular lattice

= Boolean alg. $\setminus$ distributive law

Boo. alg. = distributive lattice

$\mathbb{L}(\mathcal{H}) = \{$ closed subspaces of $\mathcal{H}\}$ ~~te~~

is a non-distributive lattice. "quantum logic"

analogy, ~ terminology : A, B, C, "propositions"

"yes-no questions"

P_A .