Rep. Analysis 2

Umgebung vs. offene Menge

$\subset$ vs $\subseteq$

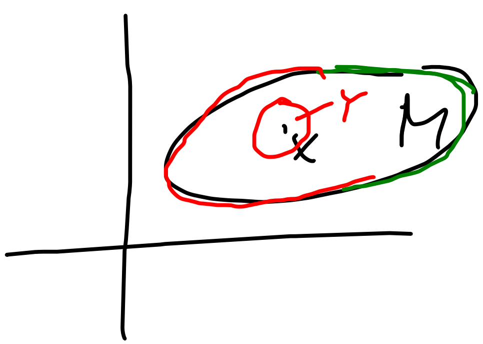
Konv. 1: $\subseteq$ Teilmenge
$\subset$ echte Teilmenge

Konv. 2: $\subset$ Teilmenge
$\subsetneq$ echte T.

X top. R., $M \subset X$, $x \in M$

M Umg von x $\overset{\text{def}}{\iff}$ $\exists$ offene Menge $Y \subset M$:
$x \in Y$

X metr. R. $\iff$ $\exists\, \varepsilon > 0 : B_\varepsilon(x) \subset M$



„offene Umg."

Topologie $\mathcal{T} \subset \mathcal{P}(X)$

$X \in \mathcal{T}, \quad \emptyset \in \mathcal{T}$

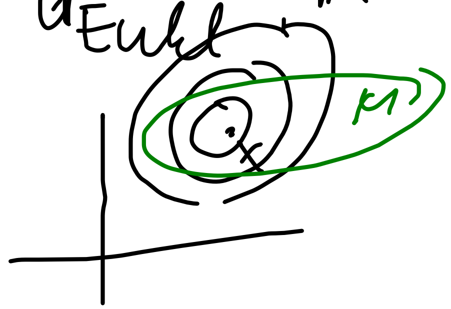
Bsp $\mathcal{T} = \{\emptyset, X\}$ ist eine Topologie.

Def Topologie: i) $X \in \mathcal{T}, \ \emptyset \in \mathcal{T}$

ii) $U, V \in \mathcal{T} \Rightarrow U \cap V \in \mathcal{T}$

iii) $U_i \in \mathcal{T} \ \forall i \in J \Rightarrow \bigcup_{i \in J} U_i \in \mathcal{T}$

Bsp d_{Eukl}, $\mathbb{R}^n$



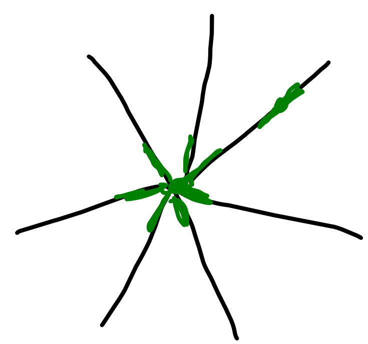
Topologie zu einem metr. R. X

$$M \in \mathcal{T} \overset{\text{Def}}{\Longleftrightarrow} \forall x \in M \ \exists \varepsilon > 0 : B_\varepsilon(x) \subset M.$$

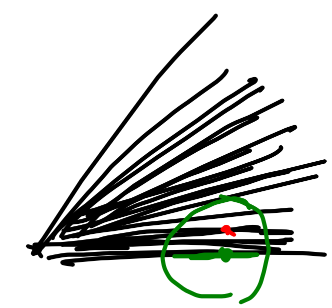
$$B_\varepsilon(x) = \{ y \in X : d(x,y) < \varepsilon \}.$$

Bspe

a)



X



b) $\{ f : [a,b] \to \mathbb{R} \mid f \text{ beschr.} \} = X$

$(X, \|\cdot\|_\infty)$ normierter Raum

Skalarprod. $\Rightarrow$ Norm $\Rightarrow$ Metrik $\Rightarrow$ Topologie

$\|x\| = \sqrt{\langle x, x \rangle}$

$$\|f\|_2 = \left(\int_a^b dx \; |f(x)|^2 \right)^{1/2}$$

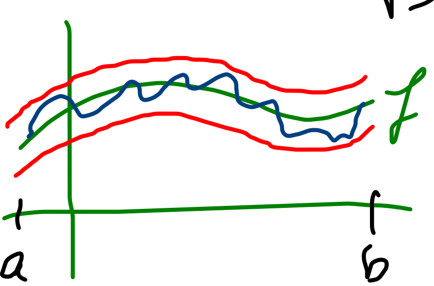
$$= \sqrt{\langle f, f \rangle}$$

für $\langle f, g \rangle = \int_a^b dx \; \overline{f(x)} \; g(x)$

$$\sum_{i=1}^{n} x_i y_i$$

$(X, \|\cdot\|_\infty)$ wie oben

$$B_\varepsilon(f) = \left\{ g : [a,b] \to \mathbb{R} \;\middle|\; g \text{ beschr. } \|f-g\|_\infty < \varepsilon \right\}$$



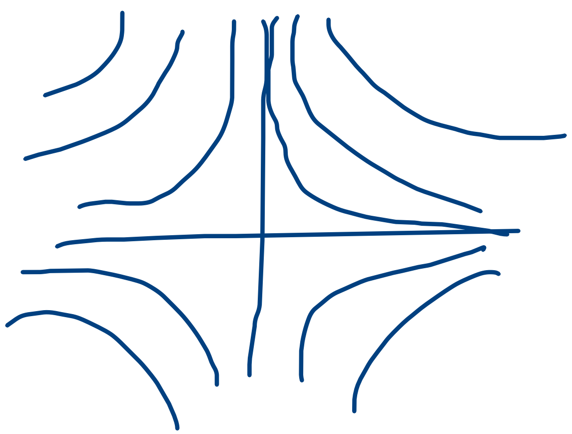
$$\sup_x |f(x) - g(x)| < \varepsilon$$

Niveaulinie vs. Niveaumenge

$f : D \to \mathbb{R}$

Niveaumenge $N_z = \{ x \in D \mid f(x) = z \}$

$D \subset \mathbb{R}^2$, N_z in der Regel eine Kurve ("Niveaulinie")

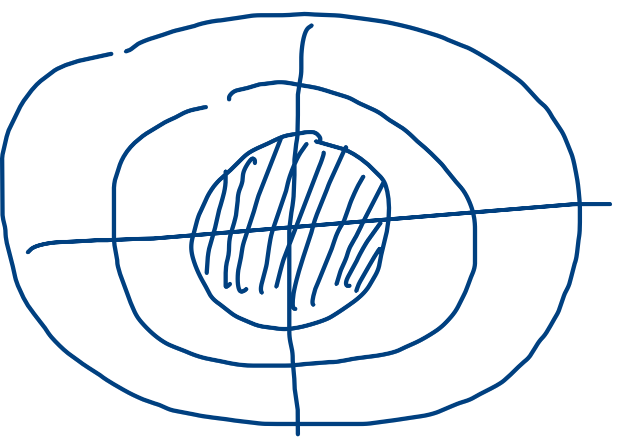


$e^{xy} = f(x,y)$

Im allg. mögl.: $N_z = \emptyset$,

$N_z = \{a\}$ Bsp $f(x,y) = x^2 + y^2$

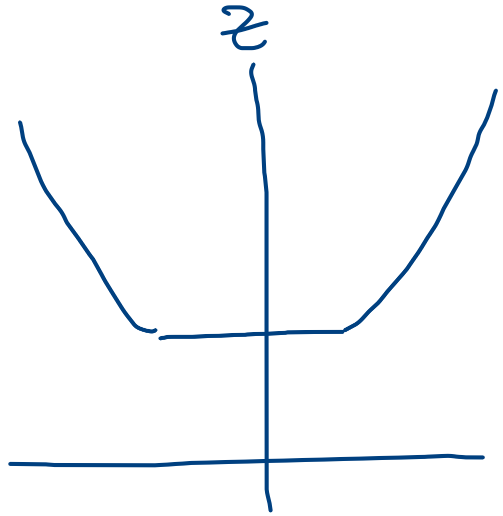
$z = 0$, $N_z = \{(0,0)\}$



$N_z = \mathbb{R}^2$ Bsp $f(x,y) = 5$

$z = 5$, $N_z = \mathbb{R}^2$

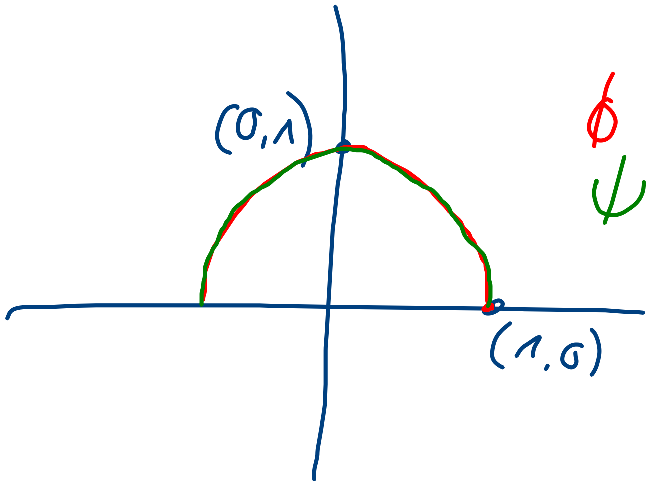
$N_z = \overline{B_1(0)}$ Bsp $f(x,y) = \max(1, x^2 + y^2)$



Kurvenlänge

$x(t), \quad x : [a,b] \longrightarrow \mathbb{R}^n$

$$L(x) = \int_a^b \left\| \frac{dx}{dt} \right\| dt$$



Bsp 2 parametrisierte Kurven:

$\phi : [0, \pi] \longrightarrow \mathbb{R}^2, \quad \phi(t) = (\cos t, \sin t)$

$\psi : [-1, 1] \longrightarrow \mathbb{R}^2, \quad \psi(s) = (s, \sqrt{1-s^2})$

Aufgabe: Berechnen Sie die Kurvenlänge.

$$L(x) = \int_a^b \left\| \frac{dx}{dt} \right\| dt$$

$$\frac{d\phi}{dt} = (-\sin t, \cos t), \quad \left\| \frac{d\phi}{dt} \right\| = \sin^2 t + \cos^2 t = 1.$$

$$L(\phi) = \int_0^\pi dt = \pi$$

$$\frac{d\psi}{ds} = \left(1, -\frac{\cancel{2}s}{\sqrt{1-s^2}}\right), \quad \left\| \frac{d\psi}{ds} \right\| = \sqrt{1 + \frac{\cancel{4}s^2}{1-s^2}} = \sqrt{\frac{1 + \cancel{3s^2}}{1-s^2}}$$

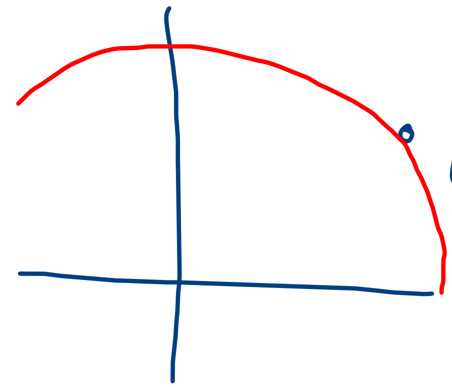
$$L(\psi) = \int_{-1}^{1} \sqrt{\frac{1+\cancel{3s^2}}{1-s^2}}\, ds.$$

$$u'v \longrightarrow u v'$$

$$\frac{1}{\sqrt{1-s^2}}$$

Polarkoord.

$(x, y) \rightsquigarrow (r, \varphi)$



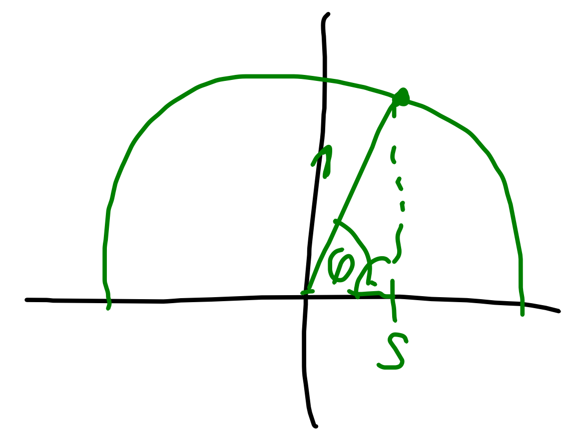
$$\phi(t) = (x(t), y(t)) = (\cos t, \sin t)$$

$$(r(t), \varphi(t)) = (1, t)$$

$$x = r \cos \varphi$$
$$y = r \sin \varphi$$

$$\psi(s) = (x(s), y(s)) = (s, \sqrt{1-s^2})$$

$$(r(s), \varphi(s)) = (1, \underbrace{\arctan \frac{\sqrt{1-s^2}}{s}}_{\arccos s})$$

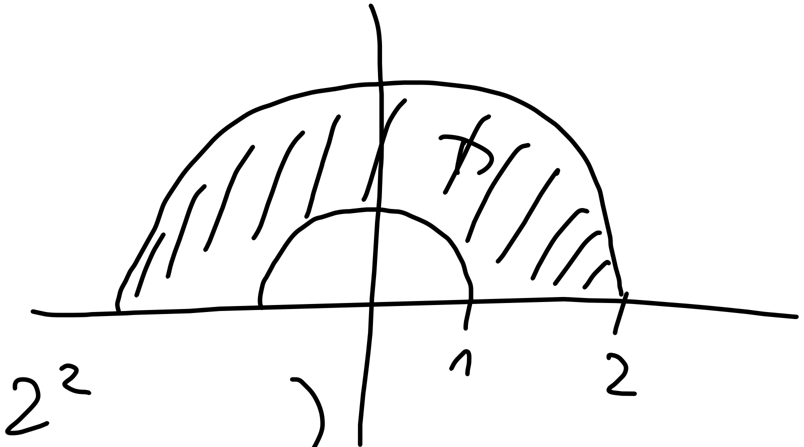


$s = \cos \varphi$

$$r(s) = \sqrt{x(s)^2 + y(s)^2} = \sqrt{s^2 + 1 - s^2} = 1.$$

$$\varphi(s) = \arctan \frac{y(s)}{x(s)} = \arctan \frac{\sqrt{1-s^2}}{s}$$

Bsp zu Polarkoo.

$D =$ 

$$D = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \begin{array}{l} 1^2 < x^2 + y^2 < 2^2 \\ y > 0 \end{array} \right\}$$

$$= \left\{ (r\cos\varphi, r\sin\varphi) \in \mathbb{R}^2 \;\middle|\; \begin{array}{l} 1 < r < 2 \\ 0 < \varphi < \pi \end{array} \right\}$$

$-\pi \leq \varphi \leq \pi$

$0 \leq \varphi \leq 2\pi$

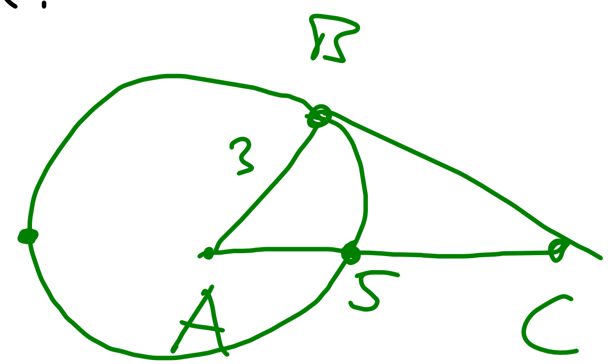
Metrik

$X = \{A, B, C\}$ metr. R.

d	A	B	C
A	0	3	5
B	3	0	x
C	5	x	0

$d(x,y) \in \mathbb{Z} \;\; \forall x,y \in X.$

Aufgabe Alle mögl. Ergänzungen?



Dreiecksungl: $\quad x \leq 8$

$$x = d(B,C) \leq d(B,A) + d(A,C) = 3+5 = 8$$

$$0 \leq x, \qquad 2 \leq x$$

$$\Uparrow$$

$$5 = d(A,C) \leq d(A,B) + \underbrace{d(C,B)}_{x} = 3 + x$$

Fazit: $x \in \{2,3,4,5,6,7,8\}$.