# Vektorwertige Integrale einer Variablen

**Satz und Def** Sei $V$ VR, $\dim V = n < \infty$.

$\underline{x} : [a,b] \to V$ st., $\mathcal{B} = (b_1 \dots b_n)$ Basis von $V$,

$\underline{x}(t) = \sum_{i=1}^{n} x_i(t)\, b_i$. Dann ist $x_i : [a,b] \to \mathbb{R}$ st., und

$$\int_a^b \underline{x}(t)\,dt := \sum_{i=1}^{n} \left( \int_a^b x_i(t)\,dt \right) b_i \in V$$

ist unabh. von der Wahl von $\mathcal{B}$.

**Bew** $x_i$ st. $\Leftarrow$ $\begin{array}{c} \underline{v} \mapsto v_i \\ V \to \mathbb{R} \end{array}$ ist lin. und (weil $\dim V < \infty$) st.

Sei $\widetilde{\mathcal{B}} = (\tilde{b}_1 \dots \tilde{b}_n)$ auch Basis von $V$.

$$\underline{x}(t) = \sum_{j=1}^{n} \tilde{x}_j(t)\, \tilde{b}_j\,, \qquad \tilde{b}_j = \sum_{i=1}^{n} a_{ij}\, b_i$$

$$\Rightarrow \underline{x}(t) = \sum_{i,j=1}^{n} \tilde{x}_j(t)\, a_{ij}\, b_i \quad \Rightarrow \quad \underline{x_i(t) = \sum_{j=1}^{n} \tilde{x}_j(t)\, a_{ij}}$$

Also $$\sum_j \left( \int_a^b \tilde{x}_j(t)\,dt \right) \tilde{b}_j = \sum_{ij} \left( \int_a^b \tilde{x}_j(t)\,dt \right) a_{ij}\, b_i$$

$$= \sum_i \left( \sum_j a_{ij} \int_a^b \tilde{x}_j(t)\,dt \right) b_i$$

$$= \sum_i \left( \int_a^b \Big[ \underbrace{\sum_j a_{ij}\, \tilde{x}_j(t)}_{x_i(t)} \Big]\, dt \right) b_i = \sum_{i=1}^{n} \left( \int_a^b x_i(t)\,dt \right) b_i.$$

$\square$

**Def** Sei $G \subset \mathbb{R}^n$ offen, $\underline{f} : G \to \mathbb{R}^n$ st. VF,

$\underline{\gamma} : [a,b] \to G$ ein st. diffbarer Weg. Dann heißt

$$\int_{\underline{\gamma}} \underline{f}(\underline{x}) \cdot d\underline{x} = \int_a^b \underline{f}(\underline{\gamma}(t)) \cdot \frac{d\underline{\gamma}}{dt}\, dt$$

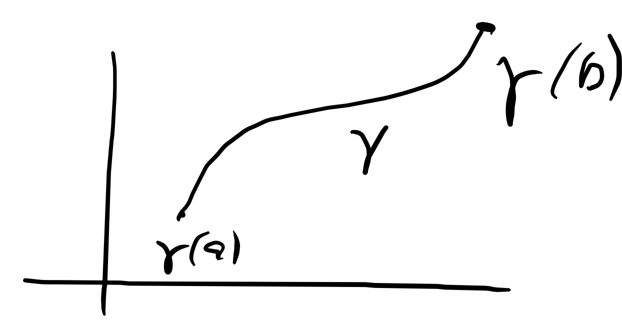
das <u>Kurvenintegral</u> oder <u>Wegintegral</u> von $\underline{f}$ über $\underline{\gamma}$.

**Bsp** $\underline{f}$ = Kraftfeld, $\int_{\underline{\gamma}} \underline{f} \cdot d\underline{x}$ = verrichtete Arbeit

**Satz 4.42** (Hauptsatz für Wegintegrale)

Wenn $\underline{f} = \operatorname{grad} F$, $F \in C^1(G, \mathbb{R})$, dann

$$\int_{\underline{\gamma}} \underline{f} \cdot d\underline{x} = F(\underline{\gamma}(b)) - F(\underline{\gamma}(a))$$



Bew $F \circ \underline{\gamma}$ ist st. diffbar,

$$\frac{d}{dt} F(\underline{\gamma}(t)) = \nabla F(\underline{\gamma}(t)) \cdot \dot{\underline{\gamma}}(t)$$

$$\Rightarrow \int_{\underline{\gamma}} \underline{f} \cdot d\underline{x} = \int_a^b \underline{f}(\underline{\gamma}(t)) \cdot \dot{\underline{\gamma}}(t)\, dt$$

$$= \int_a^b \frac{d}{dt} F(\underline{\gamma}(t))\, dt \overset{HS}{=} F(\underline{\gamma}(b)) - F(\underline{\gamma}(a)).$$

$\square$

Korollar 4.42: gilt auch noch

für $F \in C^1(G, \mathbb{R}^m)$

Korollar 4.43: (Mittelwertsatz) Für $\underline{\gamma}(t) = \underline{x} + t\underline{h}$, $t \in [0,1]$

erhält man: Wenn $\gamma([0,1]) \subset G$, dann

$$F(\underline{x}+\underline{h}) - F(\underline{x}) = \left( \int_0^1 \nabla F(\underline{x}+t\underline{h})\, dt \right) \cdot \underline{h}$$

**Bem** Formel 4.43 gilt auch noch, wenn $F$ $\mathbb{R}^m$-wertig ist, aber für $m \geq 2$:

$$\nexists\, \vartheta \in [0,1]: \cancel{\int_0^1 DF(\underline{x}+t\underline{h})\,dt \leq} \; F(\underline{x}+\underline{h}) - F(\underline{x}) = \sum_j \partial_j F(\underline{x}+\vartheta \underline{h})\, h_j .$$

**Korollar 4.46** (Schrankensatz)

Sei $G \subset \mathbb{R}^n$ offen, $F \in C^1(G, \mathbb{R}^m)$, $\underline{x}, \underline{y} \in G$

$\underline{\gamma}: [0,1] \to G$ st. diffbar mit $\underline{\gamma}(0) = \underline{x}$, $\underline{\gamma}(1) = \underline{y}$,

$$M := \sup \left\{ \left\| DF|_{\gamma(t)} \right\| : t \in [0,1] \right\} < \infty$$

[ $\|A\|$ = kleinste Lipschitz-Konstante $= \sup_{v \neq 0} \frac{\|Av\|}{\|v\|}$ Operatornorm ]

$$\| F(\underline{y}) - F(\underline{x}) \| \leq M \, L(\gamma) = M \int_0^1 \|\dot{\underline{\gamma}}(t)\|\, dt$$

Für den Beweis benötigen wir:

**Lemma 4.47** "Dreiecksungl. für Int."

Sei $\beta : [a,b] \to \mathbb{R}^n$. Dann

$$\left\| \int_a^b \beta(t)\,dt \right\| \leq \int_a^b \|\beta(t)\|\,dt$$

Bew.: $I := \int_a^b \beta(t)\,dt \in \mathbb{R}^n$

$$\|I\|^2 = \sum_{j=1}^n I_j^2 = \sum_{j=1}^n \left( \int_a^b \beta_j(t)\,dt \right) I_j$$

$$= \int_a^b \left( \sum_j \beta_j(t)\, I_j \right) dt$$

CS: $|\underline{u} \cdot \underline{v}| \leq \|\underline{u}\|\,\|\underline{v}\|$

$$\overset{CS}{\leq} \int_a^b \|\beta(t)\|\,\|I\|\,dt = \|I\| \int_a^b \|\beta(t)\|\,dt$$

$\Rightarrow$ Beh. $\square$

## Beweis des Schrankensatzes :

$$\| F(\underline{y}) - F(\underline{x}) \| \overset{4.42}{=} \Big\| \int_{\underline{\gamma}} DF(\underline{x}) \cdot d\underline{x} \Big\|$$

$$= \Big\| \int_0^1 DF\big|_{\underline{\gamma}(t)} \, \underline{\dot\gamma}(t) \, dt \Big\|$$

$$\overset{4.47}{\leq} \int_0^1 \Big\| DF\big|_{\underline{\gamma}(t)} \, \underline{\dot\gamma}(t) \Big\| \, dt$$

[ Op. norm : $\|AB\| \leq \|A\| \, \|B\|$ ]

$$\leq \int_0^1 \Big\| DF\big|_{\underline{\gamma}(t)} \Big\| \, \| \underline{\dot\gamma}(t) \| \, dt$$

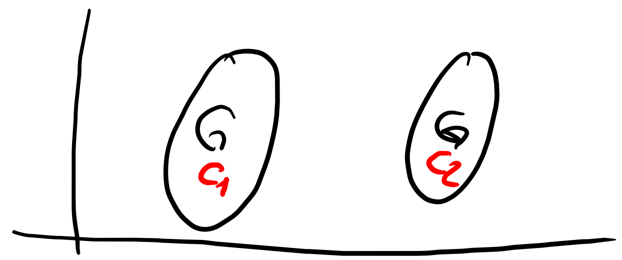
$$\leq \int_0^1 M \, \| \underline{\dot\gamma}(t) \| \, dt \leq M \int_0^1 \| \underline{\dot\gamma}(t) \| \, dt = M L(\gamma) . \quad \square$$

**Korollar** ~~Satz~~ **4.48**

Sei $G \subset \mathbb{R}^n$ Gebiet, $F \in C^1(G, \mathbb{R}^m)$,

$DF = 0$ auf $G$. Dann ist $F$ konstant.

**Bem** $G$ nicht wegzsh. $\Rightarrow F$ lok konst.




**Beweis** Seien $x, y \in G$ bel.; $G$ wegzus.

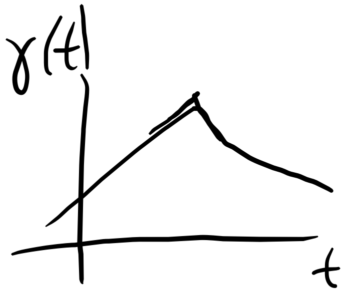
$\Rightarrow \exists \gamma : [0,1] \to G$ st. mit $\gamma(0) = x$, $\gamma(1) = y$.

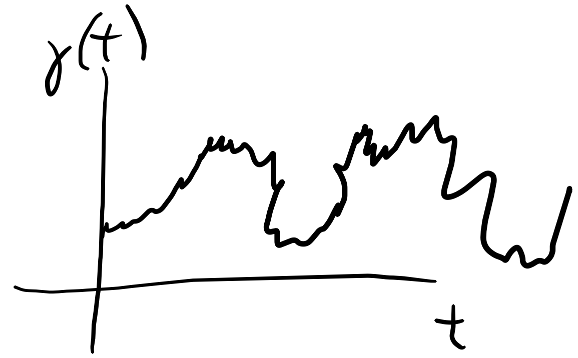
1) Falls $\gamma$ st. diffbar $\xRightarrow{\text{Schrankensatz}}$

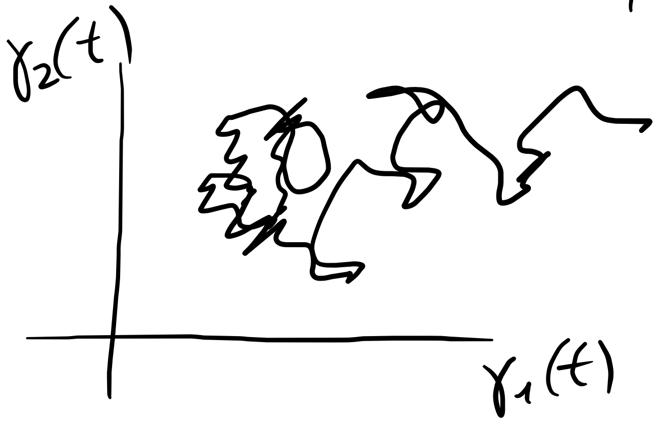
$$\|F(y) - F(x)\| \le \underbrace{M}_{= \sup \|DF\| = 0} \int_0^1 \|\gamma'(t)\| \, dt = 0$$

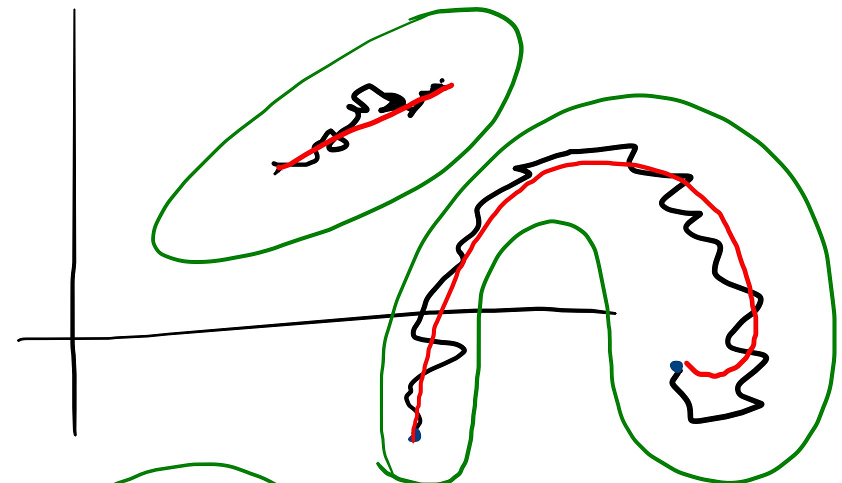
also $F(y) = F(x)$.

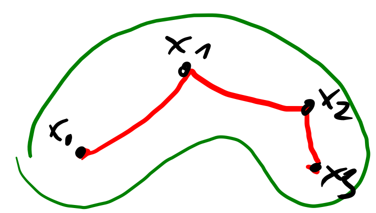
$\gamma$ st.






2) $\gamma$ nur st.:

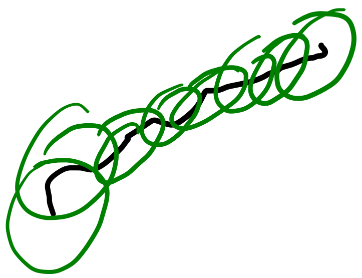
Wir zeigen: $\exists\, x = x_0, x_1, x_2, \ldots, x_k = y$, alle $x_j \in G$

$\forall j$ $\exists\, \gamma_j$ st. diffbar Kurve von $x_{j-1}$ nach $x_j$

$$\overset{1)}{\Longrightarrow} \quad F(x_{j-1}) = F(x_j) \quad \forall j \Rightarrow F(x) = F(x_1) = F(x_2) = \ldots = F(y).$$

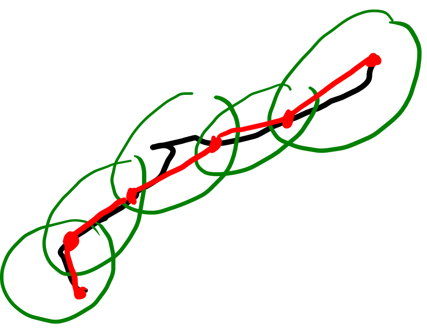
Konstruktion: $G$ offen $\Rightarrow \forall z \in G \; \exists \varepsilon(z) > 0 : B_{\varepsilon(z)}(z) \subset G$.

$\Rightarrow$ offene Üb. von $\gamma([0,1]) \subset \bigcup_{t \in [0,1]} B_{\varepsilon(\gamma(t))}(\gamma(t))$



$\gamma([0,1]) \subset G$ ist komp., weil $[0,1]$ komp.

$\Rightarrow \exists$ endl. Teilüb. $\gamma([0,1]) \subset \bigcup_{i=1}^{k} B_{\varepsilon(\gamma(t_i))}(\gamma(t_i))$



Wähle $x_j$ im Überlapp.

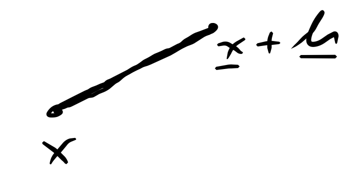
$\gamma_j =$ Strecke $[x_{j-1}, x_j] \subset B_{\varepsilon(\gamma(t_i))}(\gamma(t_i))$

weil konvex. $\square$

# Kap. 5: Taylor-Formel und lokale Extrema

1d: $\quad g: \mathbb{R} \to \mathbb{R}, \quad x, h \in \mathbb{R}$

$$g(x+h) = g(x) + g'(x)\,h + \frac{1}{2} g''(x)\,h^2 + \dots$$
$$\dots + \frac{1}{k!} g^{(k)}(x)\,h^k + o\left(|h|^k\right)$$




nd: $\quad f: \mathbb{R}^n \to \mathbb{R}, \quad \underline{x}, \underline{h} \in \mathbb{R}^n,$

$\underline{h} = t\,\underline{v}, \quad \|\underline{v}\| = 1, \quad t = \|\underline{h}\|$

$$f(\underline{x}+\underline{h}) = f(\underline{x} + t\underline{v}) =: g(t)$$
$$= f(\underline{x}) + \frac{\partial f}{\partial \underline{v}}(\underline{x})\,t + \frac{1}{2}\frac{\partial^2 f}{\partial \underline{v}^2}(\underline{x})\,t^2 + \dots$$
$$\dots + \frac{1}{k!}\frac{\partial^k f}{\partial \underline{v}^k}(\underline{x})\,t^k + o\left(|t|^k\right)$$

$$= \sum_{m=0}^{k} \frac{1}{m!} (\underline{h} \cdot \nabla)^m f(\underline{x}) + o(\|\underline{h}\|^k)$$

$$= \sum_{m=0}^{k} \frac{1}{m!} D^m f(\underline{x}) (\underline{h}, \underline{h}, \dots, \underline{h}) + o(\|\underline{h}\|^k)$$

$Df(\underline{x}) : \mathbb{R}^n \to \mathbb{R}$ lin.

$D^2 f(\underline{x}) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ bilinear

$D^3 f(\underline{x}) : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ trilinear

$D^k f(\underline{x}) : (\mathbb{R}^n)^k \to \mathbb{R}$ $k$-linear

multilinear

<u>Bem</u> $(\underline{h} \cdot \nabla)^m = \left(\sum_{j=1}^{n} h_j \partial_j\right)^m = \sum_{j_1, j_2, \dots, j_m = 1}^{n} h_{j_1} h_{j_2} \cdots h_{j_m} \partial_{j_1} \cdots \partial_{j_m}$

Bsp $\frac{\partial^2 f}{\partial \underline{v}^2}(\underline{x})\, t^2 = (\underline{h}\cdot\nabla)^2 f(\underline{x})$

$$= \sum_{j_1, j_2} h_{j_1} h_{j_2}\, \partial_{j_1}\partial_{j_2} f(\underline{x})$$

$$= \langle \underline{h}\ ,\ \mathrm{Hess} f(\underline{x})\, \underline{h} \rangle = \underline{h}\cdot\left(\mathrm{Hess} f(\underline{x})\, \underline{h}\right).$$

Also $f(\underline{x}+\underline{h}) = \sum_{m=0}^{k} \frac{1}{m!} \sum_{j_1 \dots j_m = 1}^{n} h_{j_1} \cdots h_{j_m}\, \partial_{j_1} \cdots \partial_{j_m} f(\underline{x})$

$$+ o(\|\underline{h}\|^k)$$