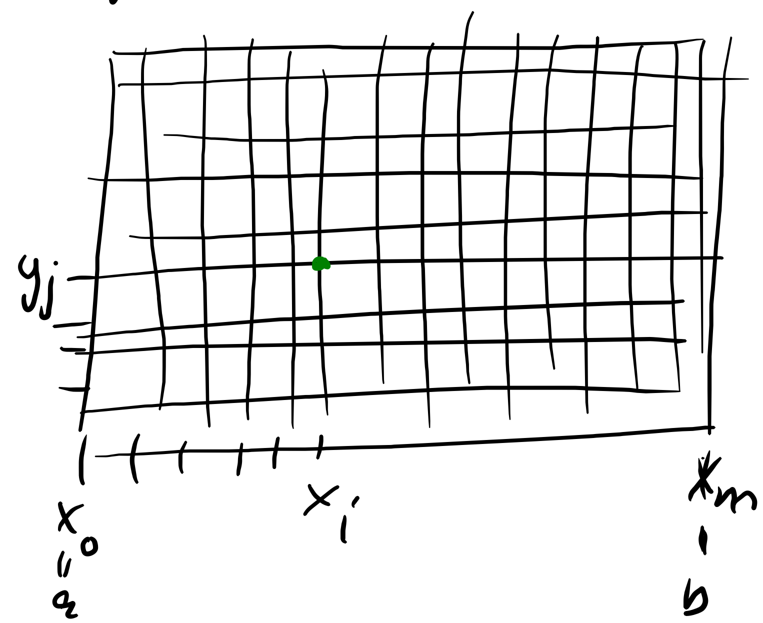
Integration in $\mathbb{R}^n$

Bsp Flächenintegral $n=2$, $Q := [a,b]\times[c,d] \subset \mathbb{R}^2$

$f : Q \to \mathbb{R}$ st.

$$\int_Q f(x,y)\,dA =$$ signiertes Vol. zw. xy-Ebene und graph(f)

$$\approx \sum_{i=1}^{m} \sum_{j=1}^{l} f(x_i,y_j)\, \underbrace{\Delta A_{ij}}_{\Delta x_i \Delta y_j}$$



$\Delta x_i = x_i - x_{i-1}$ klein

$\Delta y_j = y_j - y_{j-1}$ klein

legt nahe, dass

Def
$$\int_Q f(x,y)\,dA = \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx$$

Müssen noch zeigen:

10.5 Satz von Fubini $f : [a,b] \times [c,d] \to \mathbb{R}$ st.,

dann $$\underbrace{\int_a^b \left(\int_c^d f(x,y)\,dy \right) dx}_{\int_a^b dx \int_c^d dy\, f(x,y)} = \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy$$

Notation $$\int_a^b dx \int_c^d dy \; f(x,y)$$

Vorbereitungen

Lemma 10.2 Sei $U \subset \mathbb{R}^n$, $f : [a,b] \times U \to \mathbb{R}$ st.

$U \ni y_k \xrightarrow{k\to\infty} y_* \in U$. Die Fktfolge $f_k(x) := f(x, y_k)$

konv. glm. gegen $f_*(x) := f(x, y_*)$

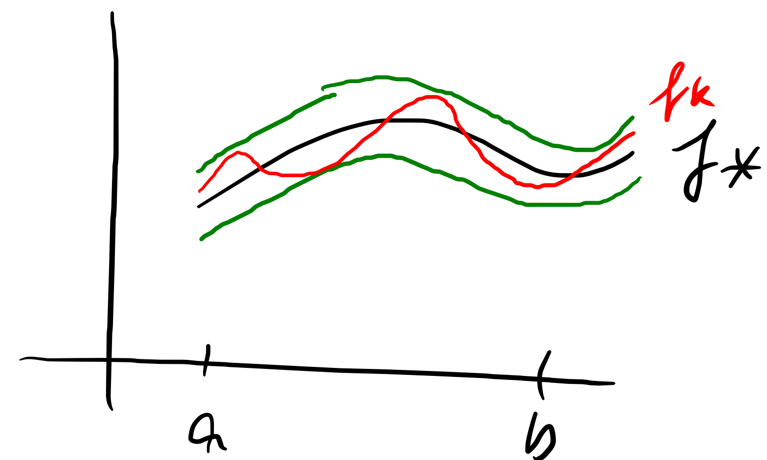
Beweis: s. Skript.

Lemma 10.3 Sei $f : [a,b] \times U \to \mathbb{R}$ st., dann ist

$$g : U \to \mathbb{R}, \qquad g(y) := \int_a^b f(x,y)\,dx \quad \text{st.}$$

Bew Allg. gilt: Wenn $f_k \to f_*$ glm., dann

$$\int_a^b f_k(x)\,dx \xrightarrow{k\to\infty} \int_a^b f_*(x)\,dx.$$



Hier sei $U \ni y_k \to y_* \in U$, also (Lemma 10.2)

$$g(y_k) = \int_a^b f(x,y_k)\,dx \xrightarrow{k\to\infty} \int_a^b f(x,y_*)\,dx = g(y_*). \quad \square$$

Lemma 10.4 Sei $U \subset \mathbb{R}^n$ offen, $f : [a,b] \times U \to \mathbb{R}$ st.,

$$\exists \ \frac{\partial f(\overset{[a,b]}{x}, \overset{U}{y})}{\partial y_j} \text{ st. } \forall j \in \{1 \dots n\}. \text{ Dann ist}$$

$$g : U \to \mathbb{R}, \qquad g(y) = \int_a^b f(x,y)\, dx \qquad C^1$$

und

$$\frac{\partial g}{\partial y_j} = \int_a^b \frac{\partial f}{\partial y_j}(x,y)\, dx.$$

Vgl. Satz von Schwarz

$$\partial_x \partial_y f = \partial_y \partial_x f$$

Hier $\int dx\, \partial_y = \partial_y \int dx$

Bew Sei $y \in U$ fest. Brauchen

$$\frac{f(x, y + h_k e_j) - f(x,y)}{h_k} \xrightarrow{k \to \infty} \frac{\partial f}{\partial y_j} \quad \text{glm in } x, y$$

für jede Nullfolge h_k in $\mathbb{R} \setminus \{0\}$

denn dann

$$\frac{g(y+h_k e_j)-g(y)}{h_k} = \int_a^b \frac{f(x,y+h_k e_j)-f(x,y)}{h_k}\,dx$$

$$\xrightarrow{k\to\infty} \int_a^b \frac{\partial f}{\partial y_j}(x,y)\,dx$$

Kriegen:

MWS : $$\frac{f(x,y+h_k e_j)-f(x,y)}{h_k} = \frac{\partial f}{\partial y_j}(x,\underbrace{y+\overset{[0,1]}{\overset{\Downarrow}{\theta}}(h_k,x,j)\,h_k e_j}_{=:\,y_k \xrightarrow[k\to\infty]{} y})$$

$\partial_{y_j} f$ st. $\overset{\text{Lemma 10.2}}{\Longrightarrow}$ glm. konv.

$\square$

Beweis des Satzes von Fubini

$$\int dx \int dy = \int dy \int dx$$

Idee: Schwarz

$$\begin{tikzcd}
f \arrow[r, "\partial_i"] \arrow[d, "\partial_j"'] \arrow[dr, phantom, "\%"] & \partial_i f \arrow[d, "\partial_j"] \\
\partial_j f \arrow[r, "\partial_i"'] & \partial_i \partial_j f
\end{tikzcd}$$

Lemma 10.4

$$\begin{tikzcd}
f \arrow[r, "\int dx"] \arrow[d, "\partial_y"'] \arrow[dr, phantom, "\%"] & g \arrow[d, "\partial_y"] \\
\partial_y f \arrow[r, "\int dx"'] \arrow[u, shift left=6ex, "\int dy"] & \partial_y \int dx \, f \arrow[u, shift right=6ex, "\int dy"']
\end{tikzcd}$$

$$\int_a^b dx \int_c^d dy\, f = \underbrace{\int_c^d dt \frac{\partial}{\partial t}}_{[\ldots]_c^d\ (HS)} \int_a^b dx \int_c^t dy\, f = \int_c^d dt \int_a^b dx \underbrace{\frac{\partial}{\partial t} \int_c^t dy\, f}_{= f(x,t)\ (HS)}$$

$$= \int_c^d dy \int_a^b dx\, f(x,y)$$

Vor HS: st.

Beweis Satz 10.5

$\tilde{g}: [a,b]\times[c,d] \to \mathbb{R}$, $\quad \tilde{g}(x,t) := \int_c^t f(x,y)\,dy$

ist st., denn

$$\left|\tilde{g}(x_1,t_1) - \tilde{g}(x_2,t_2)\right| \leq \underbrace{\left|\tilde{g}(x_1,t_1) - \tilde{g}(x_1,t_2)\right|}_{\leq |t_1-t_2| \underbrace{\sup|f|}_{<\infty} \xrightarrow{t_1\to t_2} 0} + \underbrace{\left|\tilde{g}(x_1,t_2) - \tilde{g}(x_2,t_2)\right|}_{\xrightarrow{x_1\to x_2} 0 \quad \text{Lemma 10.3}}$$

$$\text{HS} \Rightarrow \exists \frac{\partial \tilde{g}}{\partial t} = f(x,t) \text{ st.} \overset{\text{Lemma 10.4}}{\Longrightarrow}$$

$$t \mapsto \phi(t) := \int_a^b \tilde{g}(x,t)\,dx \text{ ist } C^1$$

$$\text{und } \phi'(t) = \int_a^b dx\, \frac{\partial \tilde{g}}{\partial t}(x,t) = \int_a^b dx\, f(x,t)$$

Also $\cancel{\int_a^b dx \left(\int_c^d dy \; f(x,y) \right) = \int_c^d}$

$$\int_c^d dy \left(\int_a^b dx \; f(x,y) \right) = \int_c^d dy \; \phi'(t) = \phi(d) - \underbrace{\phi(c)}_{\substack{0 \text{ weil} \\ \tilde{g}(x,c) = 0}}$$

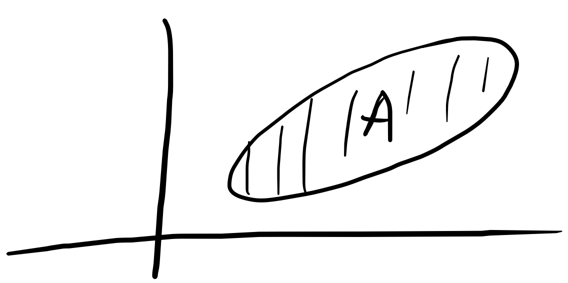
$$= \phi(d) = \int_a^b dx \left(\int_c^d dy \; f(x,y) \right)$$

$\square$

<u>Notation</u> $\quad dA = d(x,y) = dx\, dy$

<u>Andere Integrationsbereiche</u>

Wollen



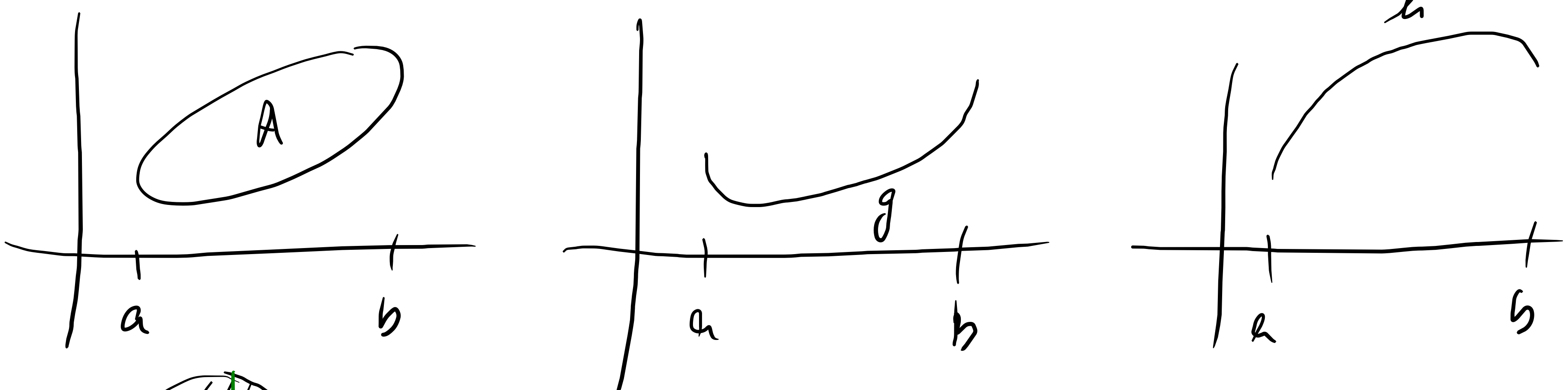
$$\int_A f(x,y)\, d(x,y)$$

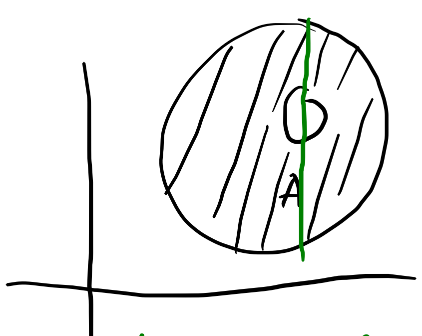
10.6 Def

$A \subset \mathbb{R}^2$ heißt <u>x-Normalbereich</u> $\Longleftrightarrow$

$\exists [a,b] \subset \mathbb{R}$ $\exists g,h : [a,b] \to \mathbb{R}$ st., st. diffbar auf (a,b)

und $A = \{ (x,y) \in \mathbb{R}^2 \mid a \le x \le b,\ g(x) \le y \le h(x) \}$





kein x-Normalbereich

oder stückweise st. diffbar

$a = a_0 < a_1 < a_2 < \dots < b = a_r$, $g, h \in C^1((a_i, a_{i+1}), \mathbb{R})$

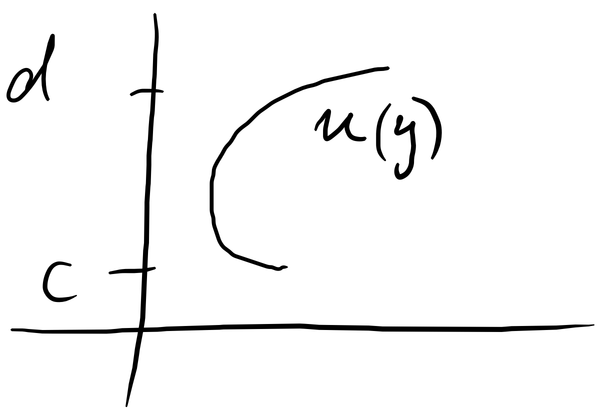
$\exists \lim\limits_{x \nearrow a_i} g(x), h(x)$

$\exists \lim\limits_{x \searrow a_i} g(x), h(x)$

A heißt <u>y-Normalbereich</u> $\iff$

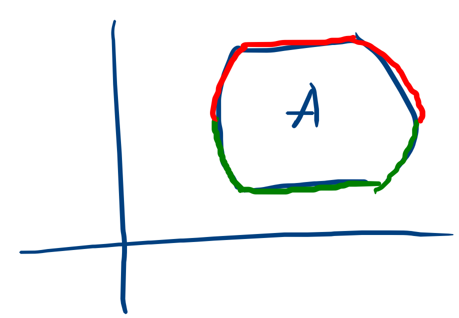
$\exists\, [c,d] \subset \mathbb{R}$ $\exists\, u, w : [c,d] \to \mathbb{R}$ st.,
stückw. st. diffbar in (c,d)

$$A = \{ (x,y) \in \mathbb{R}^2 \mid c \le y \le d,\ u(y) \le x \le w(y) \}$$

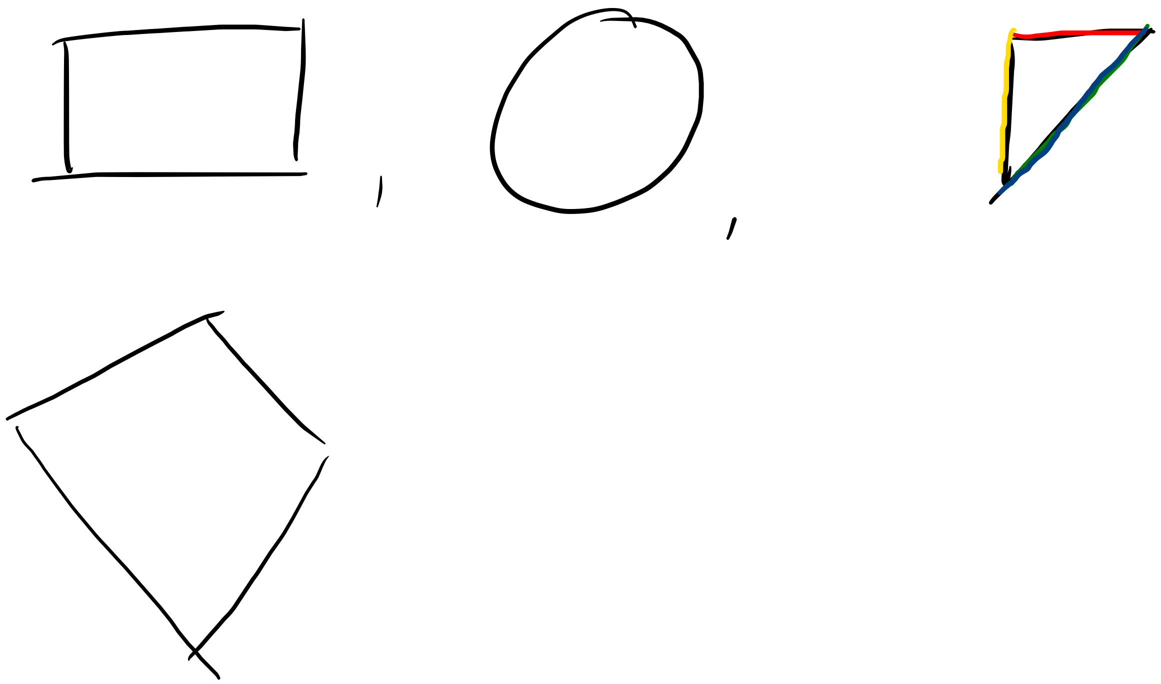


$u(c) = w(c)$, $u(d) = w(d)$ nicht verlangt.

A heißt <u>Normalbereich</u> $\iff$
A ist x-Normalbereich
und y-Normalber.



Bspe



Def 10.8 Sei $A \subset \mathbb{R}^2$, $f: A \to \mathbb{R}$ st.

Falls A x-Norm. ber., dann

$$\int_A f(x,y)\,d(x,y) := \underbrace{\int_a^b dx \left(\int_{g(x)}^{h(x)} f(x,y)\,dy\right)}_{I_x}$$

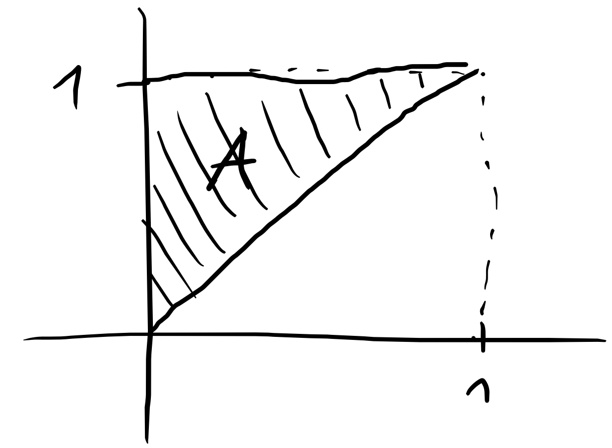
y-Norm. ber., dann

$$\int_A f(x,y)\,d(x,y) := \underbrace{\int_c^d dy \left(\int_{u(y)}^{w(y)} dx\, f(x,y)\right)}_{I_y}$$

Satz Falls A Norm. bes., dann $I_x = I_y$.

(ohne Beweis)

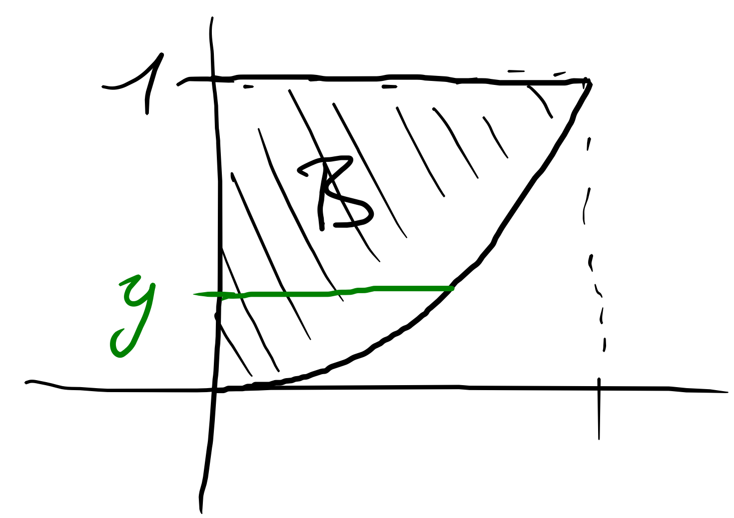
Bsp $A = \{ (x,y) \in \mathbb{R}^2 \,/\, 0 \le x \le y \le 1 \}$



$$\int_A f(x,y)\, d(x,y) = \int_0^1 dx \int_x^1 dy \; f(x,y)$$

$$= \int_0^1 dy \int_0^y dx \; f(x,y)$$

$B = \{ (x,y) \in \mathbb{R}^2 \,/\, 0 \le x \le 1,\ x^2 \le y \le 1 \}$



$$\int_B f(x,y)\, d(x,y) = \int_0^1 dx \int_{x^2}^1 dy \; f(x,y)$$

$$= \int_0^1 dy \int_0^{\sqrt{y}} dx \; f(x,y)$$