Identifizieren

Bsp $\quad A \times B = \{(a,b) \mid a \in A,\ b \in B\}$

$$A \times B \times C = \{(a,b,c) \mid a \in A,\ b \in B,\ c \in C\}$$

$$(A \times B) \times C = \{((a,b),c)\}$$

$$A \times (B \times C) = \{(a,(b,c))\}$$

wir identifizieren $((a,b),c) = (a,(b,c)) = (a,b,c)$

Ident.-Abb. $\quad i : (A \times B) \times C \to A \times B \times C$

$$i((a,b),c) = (a,b,c)$$

~~Abb~~ Allg. $i : X \to Y$, brauchen 1) bij.

2) Rechenop. erhält.

Bsp $f : \mathbb{R} \to \underbrace{\mathbb{R}^{\mathbb{R}}}_{\{g : \mathbb{R} \to \mathbb{R}\}}$, identifizieren mit $\tilde{f} : \mathbb{R}^2 \to \mathbb{R}$

$$f(x)(y) = \tilde{f}(x,y)$$

Anwendung: höhere Abl. en

$A : V \longrightarrow \mathrm{Hom}(V,W)$ linear

$i(A) : V \times V \longrightarrow W$ bilinear

$$W \ni A(\underset{V}{\underset{\cap}{v}})(\underset{V}{\underset{\cap}{u}}) = \begin{array}{l} i(A)(v,u) \\ \tilde{A}(v,u) \end{array}$$

$f : V \to W$, C^2, $Df : V \longrightarrow \mathrm{Hom}(V,W)$

$$D^2 f : V \longrightarrow \mathrm{Hom}(V, \mathrm{Hom}(V, W))$$

$$\cong \underbrace{\mathrm{Bil}(V \times V, W)}$$

$$= \{\text{bilinear Abb. en } V \times V \to W\}$$

$$D^k f : V \to \mathrm{Hom}(V, \mathrm{Hom}(V, \mathrm{Hom}(V \ldots, W))))$$

$$\cong \underbrace{\mathrm{Mult}_k(V^k, W)}$$

$$= \{\text{multilineare Abb. n } V^k \to W\}$$

$$f(x,y) = \frac{x}{y}, \quad \{(x,y) \in \mathbb{R}^2 \mid y \neq 0\} \longrightarrow \mathbb{R}$$

$$Df(x,y) = \left(\frac{1}{y}, -\frac{x}{y^2}\right), \text{ d.h. } Df(x,y)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \frac{v_1}{y} - \frac{x v_2}{y^2} \in \mathbb{R}$$

$$D^2 f(x,y)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = 0\, v_1 w_1 - \frac{1}{y^2}\, v_1 w_2 - \frac{1}{y^2}\, v_2 w_1 + 2\frac{x}{y^3}\, v_2 w_2$$

$$\text{Hess } f = \begin{pmatrix} 0 & -\frac{1}{y^2} \\ -\frac{1}{y^2} & 2\frac{x}{y^3} \end{pmatrix}$$

$$D^2 f(x,y)(v,w) = \langle v, \text{Hess } f\, w \rangle$$

$D^2 f(x,y) : V \times V \to \mathbb{R}$ bil.

Extrema unter Nebenbed.

<u>Aufgabe</u>: Zeigen Sie die AGM-Ungl. (Maclaurin 1729):

Für $a_1 \dots a_n \geq 0$ gilt

$$(a_1 \cdots a_n)^{1/n} \leq \frac{a_1 + \dots + a_n}{n}$$

<u>Anleitung</u>: Max. $f(a_1, \dots, a_n) = (a_1 \cdots a_n)^{1/n}$

unter der Nebenbed. $h(a_1 \dots a_n) = a_1 + \dots + a_n - c = 0$

<u>Lösung</u>: $\nabla f = \lambda \nabla h$ Lagr.-Gl.

$$\frac{\partial f}{\partial a_i} = \lambda \frac{\partial h}{\partial a_i}, \quad \frac{\partial f}{\partial a_i} = (a_1 \cdots a_{i-1} a_{i+1} \cdots a_n)^{1/n} \frac{1}{n} a_i^{\frac{1}{n} - 1}$$

$$= \left(\prod_{k\neq i} a_k^{1/n}\right) \frac{1}{n} a_i^{\frac{1}{n}-1}$$

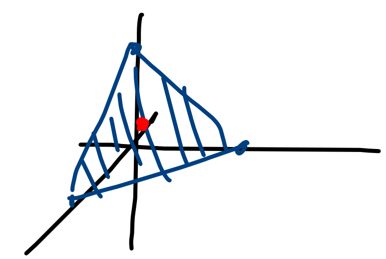
$$= \left(a_1 \cdots \widehat{a_i} \cdots a_n\right)^{1/n} \frac{1}{n} a_i^{\frac{1}{n}-1} \qquad (\;\hat{}\; = \text{auslassen})$$

$$= \frac{f}{a_i^{1/n}} \frac{1}{n} a^{\frac{1}{n}-1} = \frac{f}{n a_i}$$

$\lambda \frac{\partial h}{\partial a_i} = \lambda$, Lagr. Gl. $\frac{f}{n a_i} = \lambda \quad \forall i \in \{1 .. n\}$

$\Rightarrow a_1 = a_2 = \dots = a_n$, d. h. $a_i = \frac{c}{n}$

$\Rightarrow f(a_1 \dots a_n) = \left(\frac{c^n}{n^n}\right)^{1/n} = \frac{c}{n}$



Ziel: Max f auf $M = \{h = 0\} \cap [0,\infty)^n$ Simplex

$f|_{\partial_E M} = 0$, $f(a) > 0 \quad \forall a \in \text{Inneres}_{E=\{h=0\}}(M)$

Kein $a \in \partial_E M$ ist Max.

$\exists$ Max. weil M komp., f st.

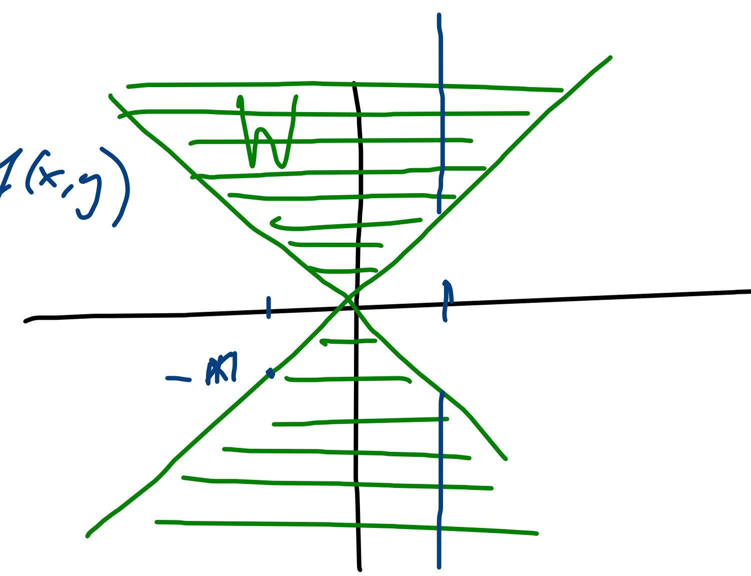
$\longrightarrow$ Max. bei $a = \left(\frac{c}{n}, \dots, \frac{c}{n}\right)$.

Integrationsgebiete

Aufgabe $W := \{(x,y) \in \mathbb{R}^2 \mid |x| \le |y|\}$

$$\int_W f(x,y)\,dA = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{-|x|} dy\; f(x,y) + \int_{-\infty}^{\infty} dx \int_{|x|}^{\infty} dy\; f(x,y)$$

$$= \int_{-\infty}^{+\infty} dy \int_{-|y|}^{|y|} dx\; f(x,y)$$

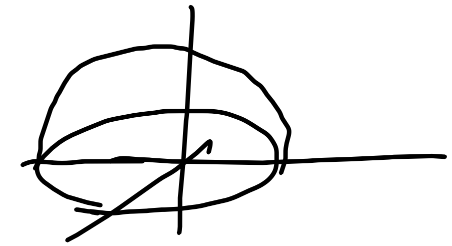


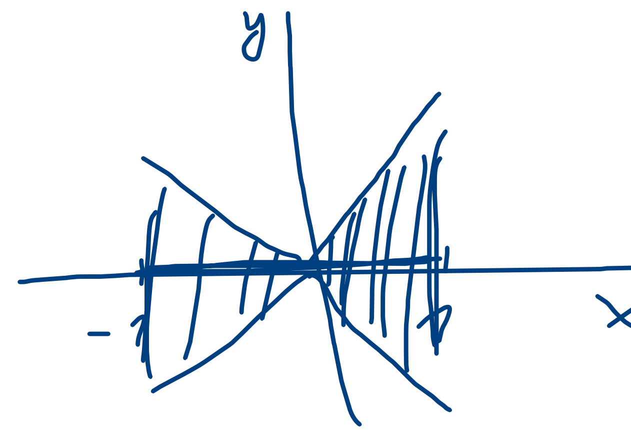
__Aufgabe__ Sei $W \subset \mathbb{R}^3$ der obere Halbball um $\underline{0}$ vom Radius 1

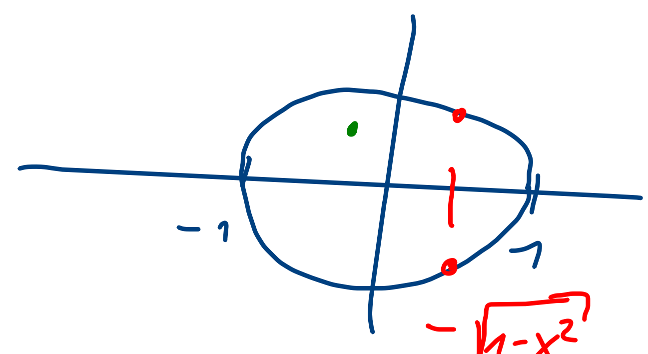
$$\int_W f(x,y,z)\, d(x,y,z)$$

$$= \int_{-1}^{1} dx \int_{\cancel{-x}\, -\sqrt{1-x^2}}^{\cancel{x}\, \sqrt{1-x^2}} dy \int_{\cancel{\emptyset}\, 0}^{\cancel{1}\, \sqrt{1-x^2-y^2}} dz \;\; f(x,y,z)$$

$$= \int_{0}^{1} dz \int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} dy \int_{-\sqrt{1-z^2-y^2}}^{\sqrt{1-z^2-y^2}} dx \;\; f(x,y,z)$$







Satz von Fubini

$f : [a,b] \times [c,d] \to \mathbb{R}$ st., dann

$$\int_a^b dx \int_c^d dy \; f(x,y) = \int_c^d dy \int_a^b dx \; f(x,y).$$

Frage: Immer noch für $f : \mathbb{R}^2 \to \mathbb{R}$ st.?

Antw.: Nein, nur wenn $\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \, |f(x,y)| < \infty$.

Bsp unendliche Matrix $(a_{ij})_{i,j \in \mathbb{N}}$

$$\begin{array}{cc}
\begin{pmatrix}
1 & -1 & 0 & 0 & 0 & 0 & \\
0 & 1 & -1 & 0 & 0 & 0 & \\
0 & 0 & 1 & -1 & 0 & 0 & \cdots \\
0 & 0 & 0 & 1 & -1 & . & \\
0 & 0 & 0 & 0 & 1 & -1 & \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \ddots
\end{pmatrix} &
\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \end{array} \\
\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & \cdots & \end{array} &
\end{array}$$

Zeilensumme $\sum_{j=1}^{\infty} a_{ij}$

Spaltensumme $\sum_{i=1}^{\infty} a_{ij}$

$$\underbrace{\sum_i \sum_j a_{ij}}_{0} \overset{?}{=} \sum_j \underbrace{\sum_i a_{ij}}_{1} \qquad 0 \neq 1$$