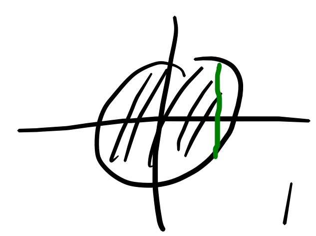
Def Normalbereich $A \subset \mathbb{R}^2$

Def 10.10 Sei A Normber.,

$$F(A) := \int_A d(x,y) \qquad (\text{d.h. } f = 1)$$

heißt die <u>Fläche von A</u>.

Bsp 10.9 $A = \overline{B_1(0)}$



$$F(A) = \int_A d(x,y) = \int_{-1}^{1} dx \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy = \int_{-1}^{1} dx \; 2\sqrt{1-x^2}$$

~~Part. Int.~~

$$\frac{d}{dx}\left(x\sqrt{1-x^2} + \arcsin x \right)$$

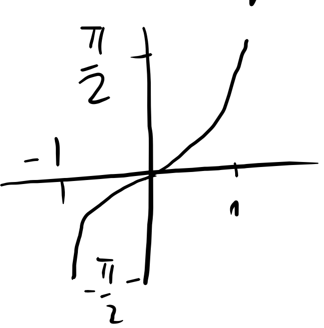
$$= \sqrt{1-x^2} + x\,\frac{1}{2}\left(1-x^2\right)^{-\frac{1}{2}}(-2x) + \frac{1}{\sqrt{1-x^2}}$$

$$\sqrt{1-x^2} = \frac{1-x^2}{\sqrt{1-x^2}}$$

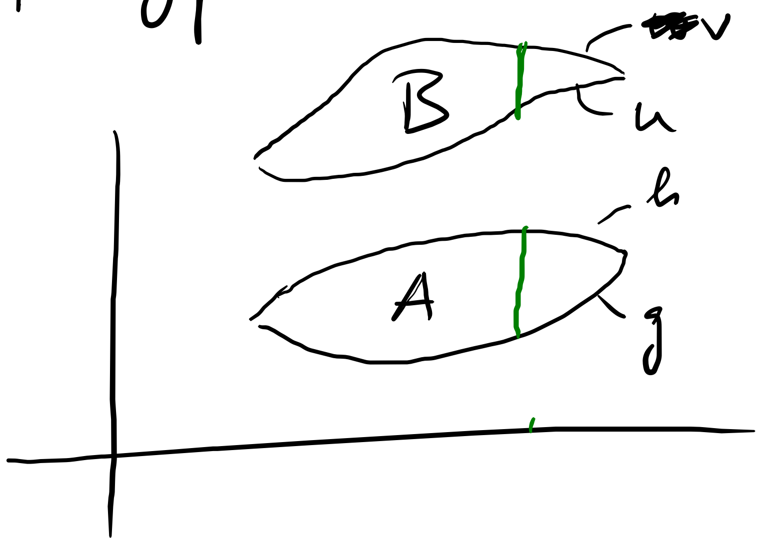
$$= \frac{1-x^2-x^2+1}{\sqrt{1-x^2}} = \frac{2\left(1-x^2\right)}{\sqrt{1-x^2}}$$

$$= 2\sqrt{1-x^2}$$

Also $F(A) = \left[x\sqrt{1-x^2} + \arcsin x \right]_{x=-1}^{x=1} = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi.$



Bem Prinzip von Cavalieri



$A = \{ g(x) \leq y \leq h(x) \}$

$B = \{ u(x) \leq y \leq v(x) \}$

Wenn $\forall x$:

$v(x) - u(x) = h(x) - g(x)$

dann $F(A) = F(B)$

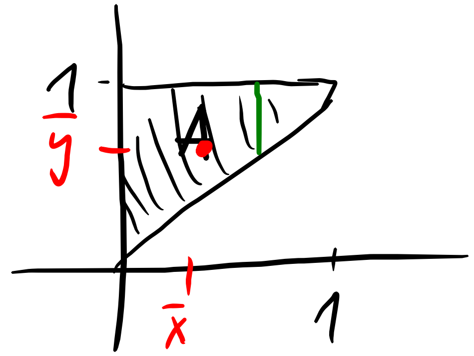
$\left(\text{Bew: } F(A) = \int_a^b dx \int_{g(x)}^{h(x)} dy = \int_a^b dx \left(h(x) - g(x) \right) = F(B). \right)$

Def Der <u>Mittelwert</u> einer Fkt $f: A \to \mathbb{R}$

ist $\bar{f} := \langle f \rangle := \frac{1}{F(A)} \int_A d(x,y) \, f(x,y)$

Def Der <u>Schwerpunkt</u> von A ist $\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}$.

Bsp



$$F(A) = \frac{1}{2}$$

$$\bar{x} = 2 \int_A d(x,y)\, x$$

$$= 2 \int_0^1 dx \int_x^1 dy\, x = 2 \int_0^1 dx\, x(1-x) = 2 \left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]_{x=0}^{x=1}$$

$$= 2\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{1}{3}.$$

$$\bar{y} = 2 \int_0^1 dx \int_x^1 dy\, y = 2 \int_0^1 dx \left[\frac{1}{2}y^2\right]_{y=x}^{y=1} = 2 \int_0^1 dx \left(\frac{1}{2} - \frac{1}{2}x^2\right)$$

$$= 2\left[\frac{1}{2}x - \frac{1}{6}x^3\right]_{x=0}^{x=1} = 2\left(\frac{1}{2} - \frac{1}{6}\right) = \frac{2}{3}$$

Ebenso in $\mathbb{R}^n$: Normalbereiche $A \subset \mathbb{R}^n$

$$g(x_1 \dots x_{n-1}) \leq x_n \leq h(x_1 \dots x_{n-1})$$

$$g, h : D \to \mathbb{R}$$

$D \subset \mathbb{R}^{n-1}$ Norm. ber.

$$\int_A d(x_1 \dots x_n)\, f(x_1 \dots x_n) = \int_D d(x_1 \dots x_{n-1}) \int_{g(x_1 \dots x_{n-1})}^{h(x_1 \dots x_{n-1})} dx_n\, f(x_1 \dots x_n)$$

Notation: $d(x_1 \dots x_n) = dV = \underset{\substack{\uparrow \\ n=2}}{dA} = d^n \underline{x} = dx$

$$\mathrm{Vol}(A) = \int_A d(x_1 \dots x_n)$$

Def 10.11 Sei $G \subset \mathbb{R}^n$, $f: G \to \mathbb{R}$

Der Träger von f (support of f) ist

$$\operatorname{supp}(f) := \overline{\{x \in G \mid f(x) \neq 0\}} \cap G$$

Allg. für $f: X \to V$, X top. R., V V.R.

$$\operatorname{supp}(f) := \overline{\{x \in X \mid f(x) \neq 0\}}$$

$$C_0(X) := \{ f \in C(X) \mid \operatorname{supp}(f) \text{ ist kompakt} \}$$

= Raum der st. Fktnen mit kompaktem Träger.

$C_0(X)$ ist V.R., denn $\operatorname{supp}(\lambda f) = \operatorname{supp}(f)$ für $\lambda \neq 0$

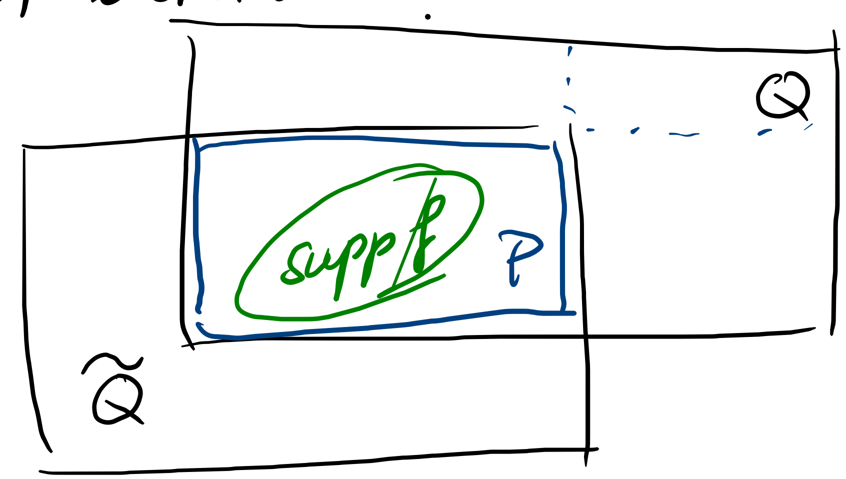
$$\underbrace{\operatorname{supp}(f+g)}_{\text{abg.}} \subset \operatorname{supp}(f) \cup \operatorname{supp}(g) \text{ komp.}$$

$\Longrightarrow$ komp. $\square$

Prop 10.12 Sei $f \in C_0(\mathbb{R}^n)$. Dann hat für jeden achsenpar. Quader $Q = [a_1, b_1] \times \cdots \times [a_n, b_n]$ mit $\operatorname{supp}(f) \subset Q$

$$\int_Q d(x_1 \dots x_n)\, f \quad \text{denselben Wert} =: \int_{\mathbb{R}^n} d(x_1 \dots x_n)\, f \,.$$

Bew Seien $Q, \tilde{Q}$ 2 Quader, die $\operatorname{supp}(f)$ enthalten. Dann ist $P := Q \cap \tilde{Q}$ ebenfalls Quader, der $\operatorname{supp}(f)$ enthält. Daher



$$\int_Q f = \int_P f = \int_{\tilde{Q}} f \qquad \square$$

Bem 10.14 $\int_{\mathbb{R}^n}$ ist invariant unter Translationen:

$$\int_{\mathbb{R}^n} d^n\underline{x}\, f(\underline{x} + \underline{a}) = \int_{\mathbb{R}^n} d^n\underline{x}\, f(\underline{x}) \qquad \forall \underline{a} \in \mathbb{R}^n \ \forall f \in C_0(\mathbb{R}^n).$$

($\Leftarrow$ entspr. in $\mathbb{R}^1$)

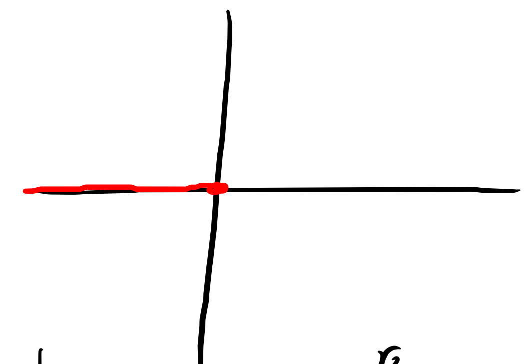
Die Transformationsformel

$=$ Substitutionsregel im $\mathbb{R}^n$

Bsp Polarkoordinaten

$$\Phi : (0,\infty) \times (-\pi, \pi) \longrightarrow \mathbb{R}^2 \setminus \{ (x_1, 0) \mid x_1 \le 0 \}$$

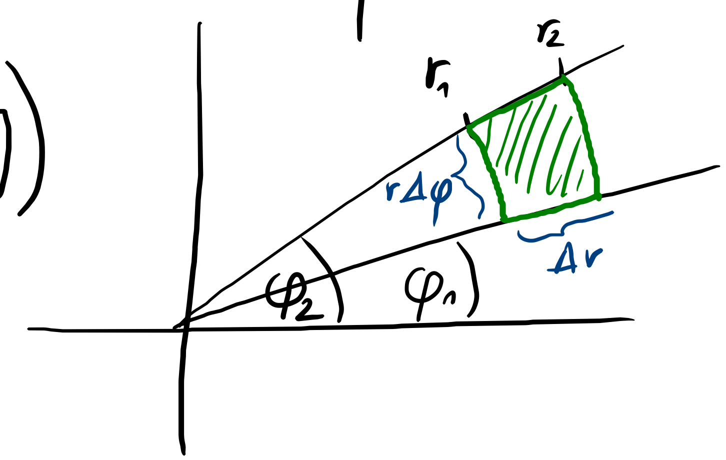
$$(r, \varphi) \longmapsto \begin{pmatrix} r \cos\varphi \\ r \sin\varphi \end{pmatrix} \quad \text{Diffeo}$$



$$\tilde{f}(r, \varphi) = f \circ \Phi$$

"polares Rechteck" $\Phi\big([r_1, r_2] \times [\varphi_1, \varphi_2]\big)$

hat die Fläche $F = \dfrac{\varphi_2 - \varphi_1}{2} \left(r_2^2 - r_1^2 \right)$



$r_2 = r + \Delta r, \; r_1 = r, \; \varphi_2 = \varphi + \Delta\varphi, \; \varphi_1 = \varphi$

$\Longrightarrow \quad F = r \, \Delta r \, \Delta\varphi + O(\Delta r^2)$

legt nahe

$$\int_{\mathbb{R}^2} d(x,y)\, f(x,y) \approx \sum_{ij} \tilde{f}(r_i, \varphi_j)\, \underbrace{\Delta A_{ij}}_{\approx r_i \,\Delta r_i \,\Delta \varphi_j}$$

$$\approx \int_{-\pi}^{\pi} d\varphi \int_{0}^{\infty} dr\; \tilde{f}(r, \varphi)\, r$$

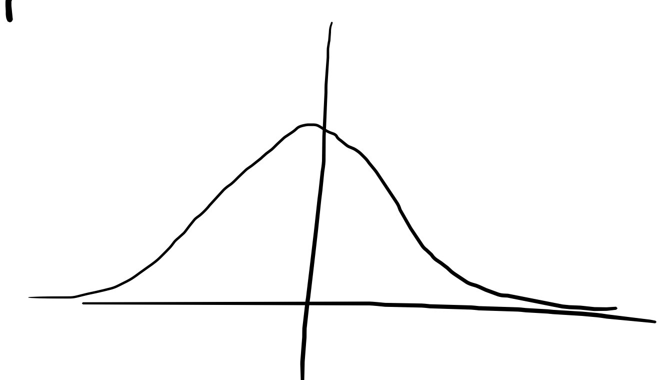
<u>Merke</u> "$d(x,y) = r\, dr\, d\varphi$"

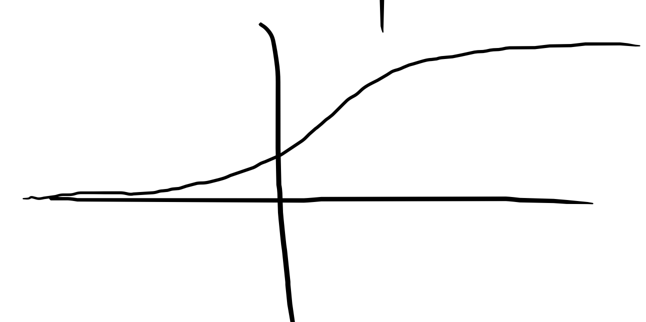
<u>Bsp</u> (benutzt $\int_{\mathbb{R}^n}$ ohne komp. Träger)

$$f(x,y) = e^{-x^2-y^2}, \quad \tilde{f}(r,\varphi) = e^{-r^2}$$

$$\int_{\mathbb{R}^2} d(x,y)\, e^{-x^2-y^2} = \int_{-\infty}^{\infty} dx\, e^{-x^2} \int_{-\infty}^{\infty} dy\, e^{-y^2}$$

$$= \left(\int_{-\infty}^{\infty} dx\, e^{-x^2} \right)^2$$





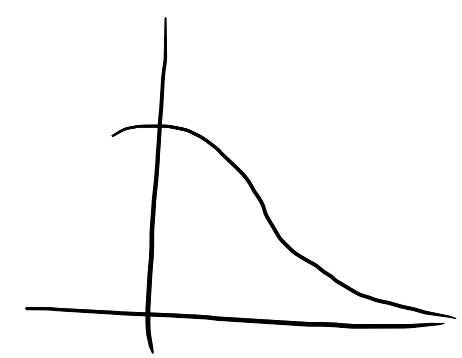
$$\int_{\mathbb{R}^2} d(x,y)\, e^{-x^2-y^2} = \int_{-\pi}^{\pi} d\varphi \int_0^\infty dr\; e^{-r^2} r$$

$$= 2\pi \int_0^\infty dr\; \underbrace{e^{-r^2} r}_{-\frac{1}{2}\frac{d}{dr} e^{-r^2}}$$

$$= -\pi \left[e^{-r^2} \right]_{r=0}^{r=\infty}$$

$$= -\pi \lim_{b\to\infty} \left[e^{-r^2} \right]_{r=0}^{r=b}$$

$$= -\pi \left(0 - 1 \right)$$



$$= \pi .$$

$$\Longrightarrow \int_{-\infty}^{\infty} dx\; e^{-x^2} = \sqrt{\pi} .$$

Transformationsformel

$\Phi : \underset{\mathbb{R}^n}{\overset{\cap}{U}} \to \underset{\mathbb{R}^n}{\overset{\cap}{G}}$ Diffeo

$$\int_G d^n\underline{y}\, f(\underline{y}) = \int_U d^n\underline{x}\, f(\Phi(\underline{x})) \cdot \text{Faktor}$$

Substitutionsregel in $\mathbb{R}^1$:

$$\int_{\Phi(a)}^{\Phi(b)} dy\, f(y) = \int_a^b dx\, f(\Phi(x)) \overbrace{\Phi'(x)}^{\text{Faktor}}$$

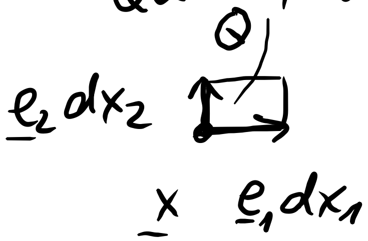
<u>Merke</u> "$dy = \Phi'(x)\,dx$"

Polarkoo: $d^n\underline{x} = dr\, d\varphi$

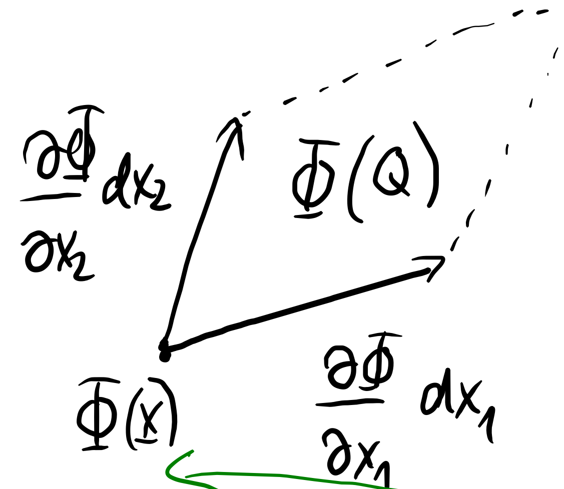
Faktor $= r$

allg. $\mathbb{R}^n$: Faktor $=$ "$\dfrac{\mathrm{Vol}\,\Phi\big([x_1, x_1+dx_1] \times \cdots \times [x_n, x_n+dx_n]\big)}{dx_1 \cdots dx_n}$"

infinitesimaler Quader, $\mathrm{Vol} = dx_1 \cdots dx_n$



$\xrightarrow{\ \Phi\ }$



$$\mathrm{Vol}\, \Phi(Q) = \left| \det \left(\frac{\partial \Phi}{\partial x_1} dx_1 , \dots , \frac{\partial \Phi}{\partial x_n} dx_n \right) \right|$$

$$= \left| \det \left(\frac{\partial \Phi}{\partial x_1} \cdots \frac{\partial \Phi}{\partial x_n} \right) \right| dx_1 \cdots dx_n$$

Faktor

$$= \underbrace{\left| \det \left(D \Phi \big|_{\underline{x}} \right) \right|}_{\substack{\text{Jacobi-Determinante} \\ = \text{Funktionaldeterminante}}} dx_1 \cdots dx_n$$

Jacobi-Matrix = Funktionalmatrix

Merke

Faktor = $\cancel{\Phi}$ Jacobi-Det.

Bsp Polarkoo

$$\Phi(r,\varphi) = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \end{pmatrix}, \quad D\Phi = \begin{pmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{pmatrix}$$

$$\det D\Phi = r\cos^2\varphi + r\sin^2\varphi = r$$