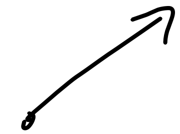
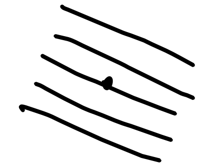
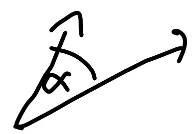
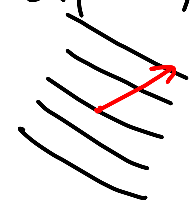
Kovektoren

graphisch: Vektor  Kovektor α 

Sk. prod: $\langle u, v \rangle = \|u\| \, \|v\| \cos\alpha = \sum_{i=1}^{n} u_i v_i$



$\alpha(v) \in \mathbb{R}$



Matrizen: Vektor $n \times 1$  Kovektor $1 \times n$ 

d.h. $v \mapsto \langle u, v \rangle$

oder $\alpha(v) = \langle u, v \rangle$

$$\alpha(v) = \begin{pmatrix} \alpha_1 & \cdots & \alpha_n \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \sum_{i=1}^{n} \alpha_i v_i$$

Sk. prod

$\alpha \longleftrightarrow u$, $\quad u_i = \alpha_i$

ONB

$\alpha = \langle u, \cdot \rangle$

$\nabla f(v) \in \mathbb{R}$, $\quad f(x+v) \simeq f(x) + \nabla f(v)$.

Dichte

z.B. Massendichte, Ladungsdichte, Wahrscheinlichkeitsdichte

$\rho : \mathbb{R}^d \longrightarrow \mathbb{R}$

$$\text{Gesamtmenge} = \int_{\mathbb{R}^d} \underbrace{d^d \underline{x}}_{dx_1\, dx_2 \cdots dx_n \,=\, d(x_1 \ldots x_n)} \; \rho(\underline{x})$$

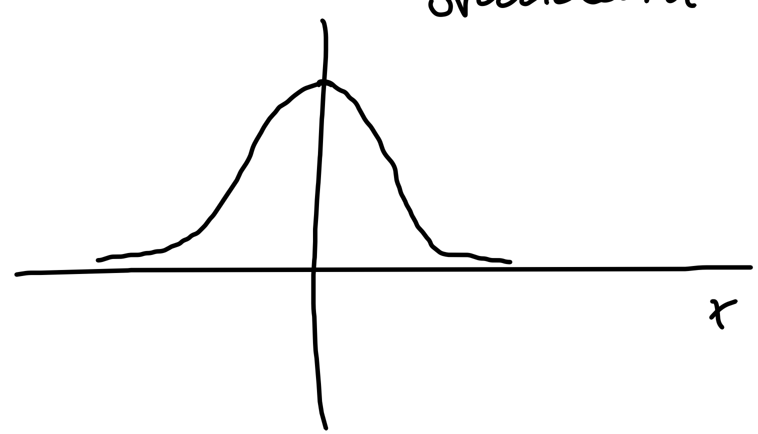
$U \subset \mathbb{R}^d$, $\text{Inhalt}(U) = \int_U d^d\underline{x}\; \rho(\underline{x})$

Bsp Gaußsche Zufallsvariable $X \sim \mathcal{N}(0,1)$

Standard-Normalverteilung

$d=1$

$$\rho(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$



$$\mathbb{P}(X \in U) = \int_U dx\; \rho(x)$$

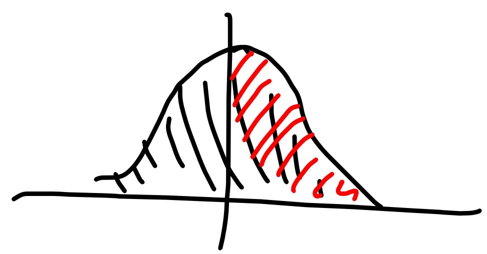
$$\text{Gesamtmenge} = \mathbb{P}(X \in \mathbb{R}) = 1$$

$$= \int_{\mathbb{R}} dx\, p(x) = \int_{-\infty}^{\infty} dx\, \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$y = x/\sqrt{2}$

$$= \int_{-\infty}^{\infty} dy\, \cancel{\sqrt{2}}\, \frac{1}{\sqrt{\cancel{2}\pi}}\, e^{-y^2} = \frac{1}{\sqrt{\pi}} \underbrace{\int_{-\infty}^{\infty} dy\, e^{-y^2}}_{\sqrt{\pi}} = 1.$$

$$\mathbb{P}(X > 0) = \frac{1}{2}$$



$$\mathbb{P}(X = 0) = 0.$$

$$\mathbb{P}(X = x) = 0 = \int_{x}^{x} dy\, p(y)$$

$U = \{x\}$

Träger

Def $\operatorname{supp}(f) = \overline{\{x \mid f(x) \neq 0\}}$

Aufgabe Bestimmen Sie den Träger folgender Fktnen auf $\mathbb{R}^2$

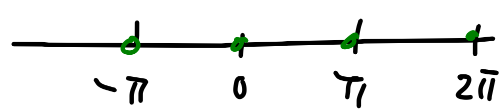
(1) $f(x,y) = \sin(x)\sin(y)$

(2) $g(x,y) = 0$

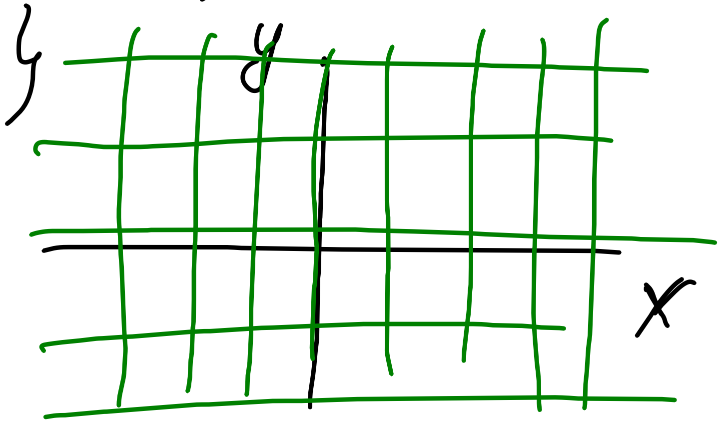
(3) $h(x,y) = \max\{f(x,y), g(x,y)\}$

(4) $k(x,y) = \max\{0, x+y\}$

Lösung (1)

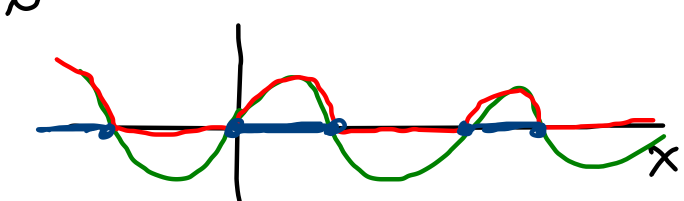


$\operatorname{supp}(f) = \mathbb{R}^2$



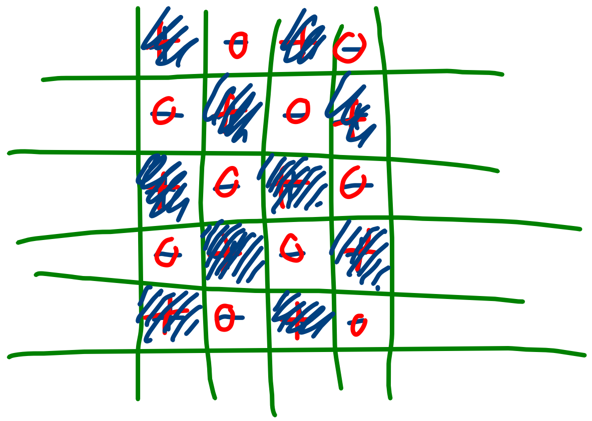
(2) $\emptyset$

(3)



$\max\{\sin(x), 0\}$

(3)



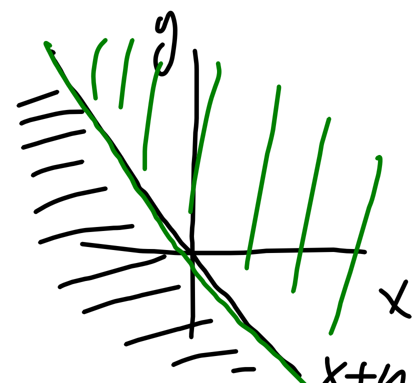
$$f(x,y) = \sin(x)\sin(y)$$

$$h = \max\{f, 0\}$$

$$\mathrm{supp}(h) = \bigcup_{\substack{m,n\in\mathbb{Z} \\ m+n \text{ gerade}}} [\pi m, \pi(m+1)] \times [\pi n, \pi(n+1)]$$

(4) $\mathrm{supp}\left(\max\{0, x+y\}\right)$

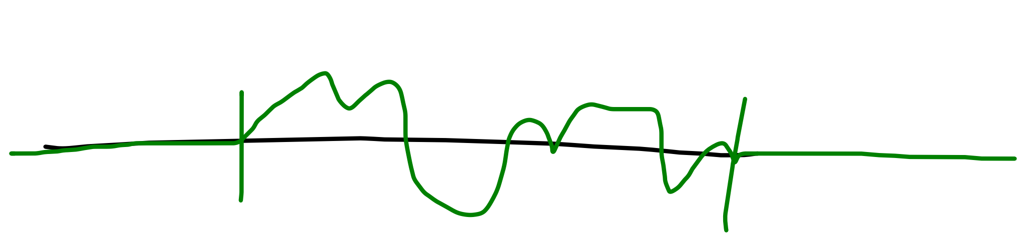
$$= \{(x,y) \in \mathbb{R}^2 \,/\, y \geq -x\}$$



$x+y=0$

d.h. $y=-x$

Kompakter Träger



Zeige: Jedes stetige $f : \mathbb{R}^2 \to \mathbb{R}$ ist

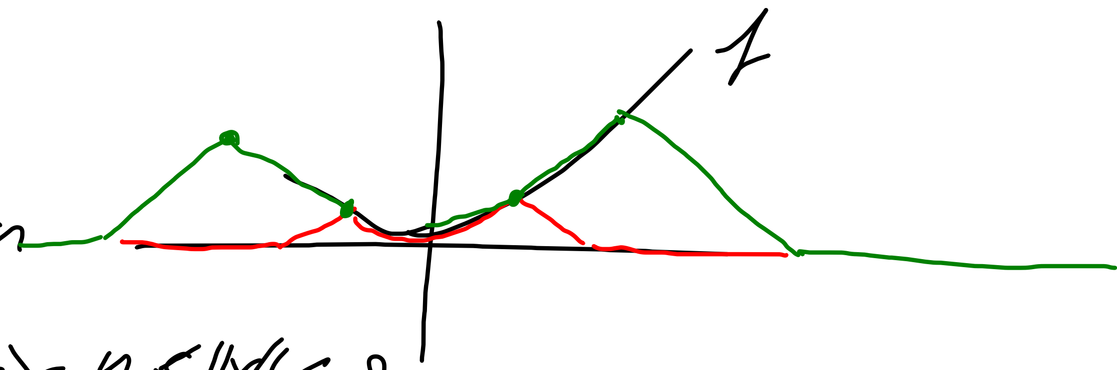
(a) punktweiser, aber

(b) nicht gleichmäßiger Limes

einer Folge $f_n \in C_0(\mathbb{R}^2, \mathbb{R})$ (komp. Träger)

supp

Lösung: (a) Setze



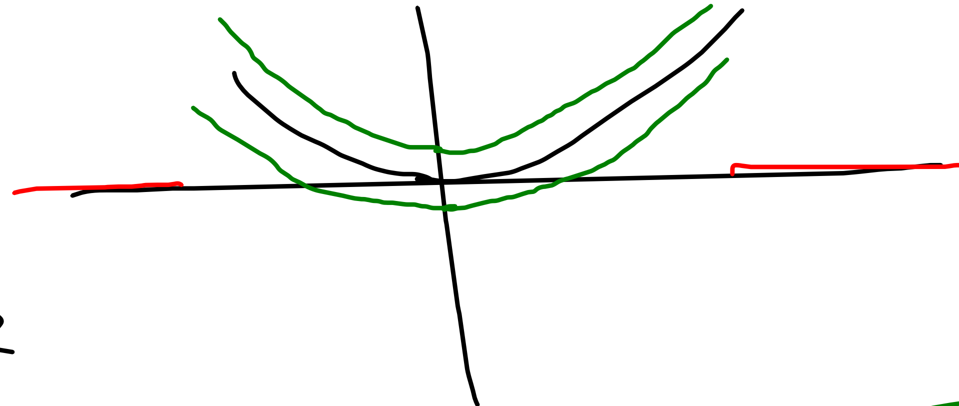
$$f_n(x_1, x_2) := f(x_1, x_2) \quad \text{für } \|x\| \le n$$

$$f_n(x_1, x_2) := f\left(\frac{n\,x}{\|x\|}\right)\left(2 - \frac{\|x\|}{n}\right) \quad \text{für } n \le \|x\| \le 2n$$

$$f_n(x_1, x_2) := 0 \quad \text{für } \|x\| > 2n$$

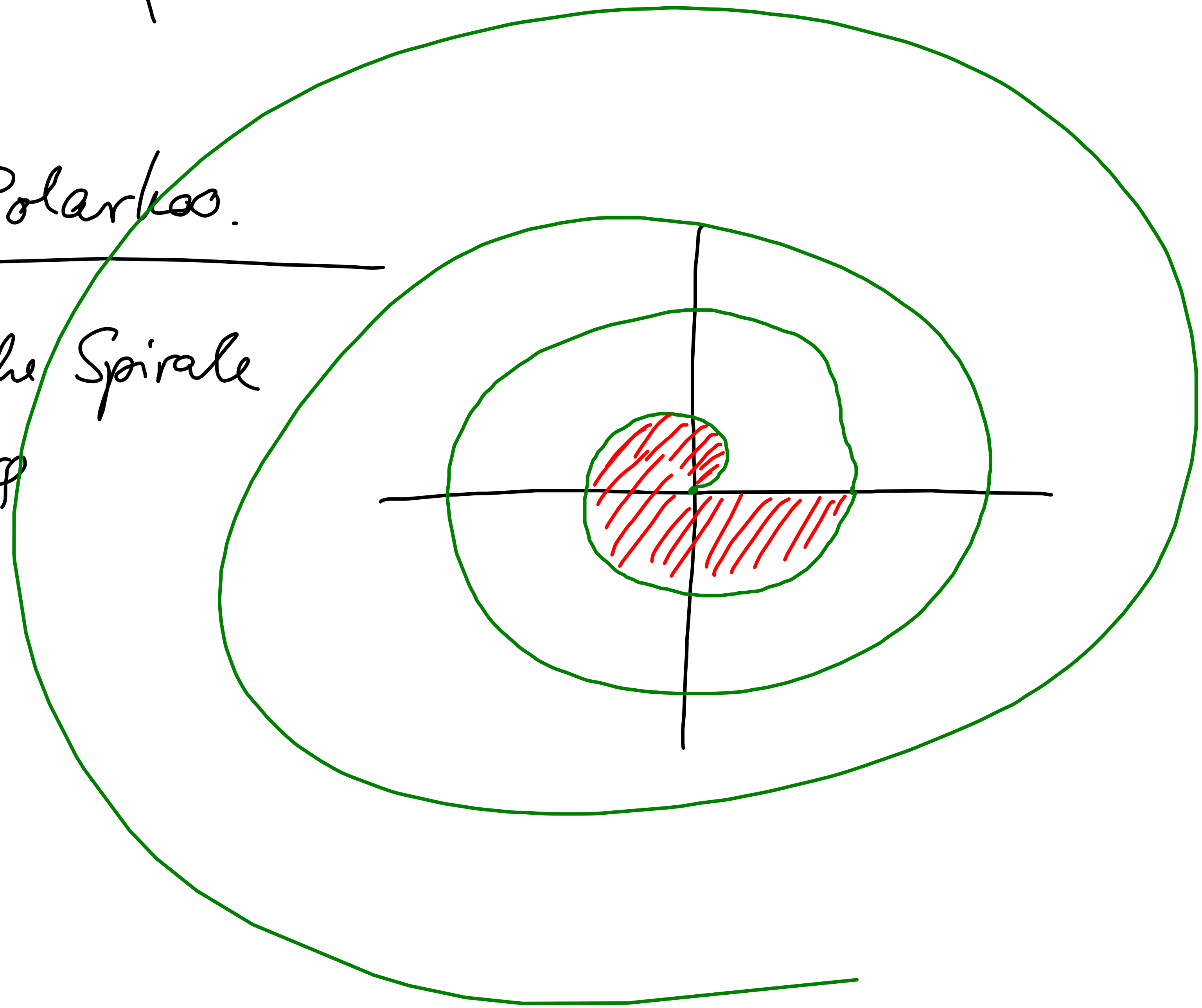
$\longrightarrow$ pktw. gegen $f(x)$

(b) glm



$f(\underline{x}) = \|\underline{x}\|^2$

Integral in Polarkoo.



Bsp Archimedische Spirale

Gl. $r = a\varphi$

$a > 0$ Konst.

$\gamma : [0,\infty) \to \mathbb{R}^2$

$$\gamma(\varphi) = \begin{pmatrix} a\varphi \cos\varphi \\ a\varphi \sin\varphi \end{pmatrix}$$

Aufgabe: Finde F.

Lösung: $A = \{ (r\cos\varphi, r\sin\varphi) \in \mathbb{R}^2 \mid 0 \le \varphi \le 2\pi, 0 \le r \le a\varphi \}$

$$F = F(A) = \int_A d(x,y)$$

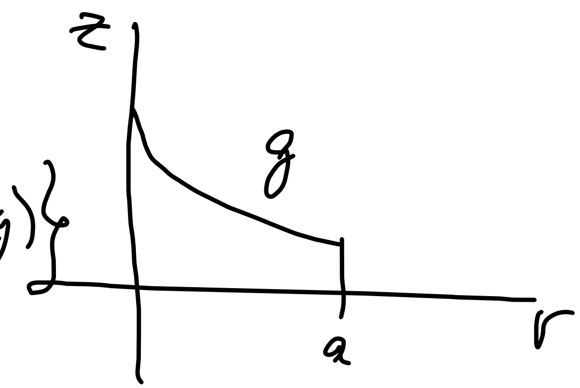
$$= \int_0^{2\pi} d\varphi \int_0^{a\varphi} dr\, r = \int_0^{2\pi} d\varphi \left[\frac{r^2}{2}\right]_{r=0}^{r=a\varphi} = \int_0^{2\pi} d\varphi\, \frac{a^2\varphi^2}{2}$$

$$= \frac{a^2}{2} \left[\frac{\varphi^3}{3}\right]_{\varphi=0}^{\varphi=2\pi} = \frac{a^2\, 8\pi^3}{6} = \frac{4}{3} a^2 \pi^3 .$$

Bsp Volumen eines Rotationskörpers: Rotation um z-Achse

Sei $g : [0,a] \to \mathbb{R}$ mit $g(r) \ge 0 \quad \forall r \in [0,a]$

setze $f(x_1, x_2) = g(\|\underline{x}\|)$ für $\|\underline{x}\| \le a$



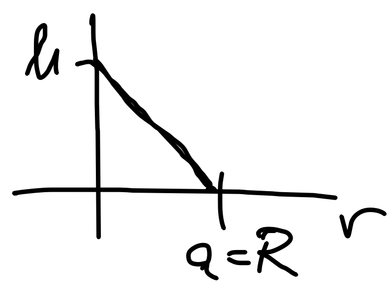
Aufgabe Finde $V := \mathrm{Vol}\{(x,y,z) \in \mathbb{R}^3 \mid 0 \le z \le f(x,y)\}$.

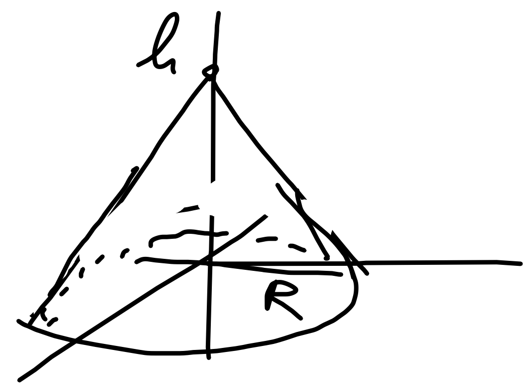
Lösung: $V = \int\limits_{\overline{B_a(0)}} d(x,y)\, f(x,y)$

$$= \int_0^{2\pi} d\varphi \int_0^a dr\, r\, \underbrace{f(r\cos\varphi, r\sin\varphi)}_{g(r)} = 2\pi \int_0^a dr\, r\, g(r)$$

Bsp Vol (Kreiskegels mit Höhe h und Radius R) = ?

Lösung: $g(r) = h - \frac{h}{R} r$





$a = R$

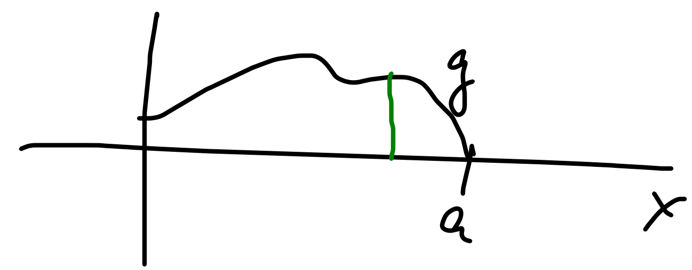
$$V = 2\pi \int_0^R dr \left(hr - \frac{h}{R} r^2\right) =$$

$$= 2\pi \left[\frac{hr^2}{2} - \frac{hr^3}{3R}\right]_{r=0}^{r=R} = 2\pi h \left(\frac{R^2}{2} - \frac{R^2}{3}\right) = \frac{\pi h R^2}{3}.$$

Normalbereiche

Bsp Volumen eines Rotationskörpers: Rotation um die x-Achse

Sei $g : [0,a] \to \mathbb{R}$ mit $g(x) \geq 0 \ \forall x$



Aufgabe: Finde

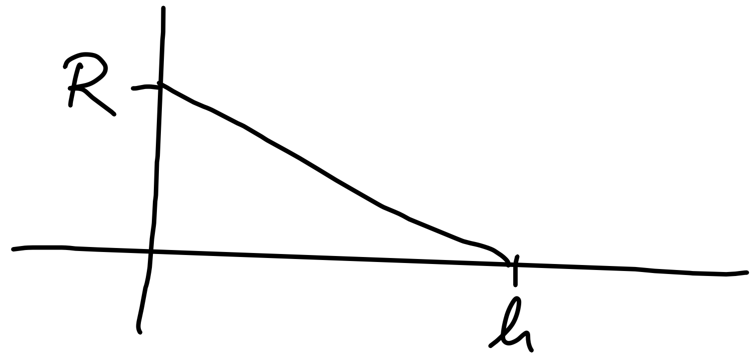
$$V = \mathrm{Vol}\left\{ (x,y,z) \in \mathbb{R}^3 \,\middle|\, \sqrt{y^2+z^2} \leq g(x),\ 0 \leq x \leq a \right\}.$$

Erinnerung: Normalber. A

$$\int_A f = \int_{x_{1min}}^{x_{1max}} dx_1 \left(\int_{\{(x_2 .. x_n) \in \mathbb{R}^{n-1} \mid (x_1, \dots, x_n) \in A\}} dx_2 \cdots dx_n \, f \right)$$

Hier $$V = \int_A 1 = \int_0^a dx \underbrace{\int_{B_{g(x)}(0)} d(y,z)} = \underline{\underline{\int_0^a dx \, \pi \, g(x)^2}}$$

Bsp Kreiskegel



$$a = h,\quad g(x) = R - \frac{R}{h}\, x$$

$$V = \int_0^h dx\, \pi \left(R - \frac{R}{h} x\right)^2 = \int_0^h dx\, \pi \left(R^2 - \frac{2R^2}{h} x + \frac{R^2}{h^2} x^2\right)$$

$$= \pi \left[R^2 x - \frac{R^2}{h} x^2 + \frac{R^2}{3h^2} x^3 \right]_{x=0}^{x=h} = \pi \left(R^2 h - \frac{R^2 h^{\cancel{2}}}{\cancel{h}} + \frac{R^2 h^{\cancel{3}}}{3 \cancel{h^2}}\right)$$

$$= \frac{\pi R^2 h}{3}.$$