Lemma von Grönwall

Sei $a<b$, $u:[a,b]\to[0,\infty)$ st.

Wenn $\exists L, C \geq 0 : \forall t \in [a,b]$:

$$u(t) \leq C + L \int_a^t u(s)\,ds$$

(insbes. wenn $\dot{u} \leq L u$ und $u(a) = C$)

dann

$$u(t) \leq C\, e^{L(t-a)}.$$

Beweis Falls $C > 0$: Setze $U : [a,b] \to \cancel{[}(0,\infty)$

$$U(t) := C + L \int_a^t u(s)\,ds$$

U ist C^1, $\dot{U} = L u(t) \geq 0 \Rightarrow U$ monoton steigend.

$U(a) = C > 0 \Rightarrow U(t) \geq C \quad \forall t \in [a,b].$

Daher $\frac{d}{dt} \ln U(t) = \frac{\dot{U}(t)}{U(t)} = \frac{L\, u(t)}{U(t)} \underset{u \le U}{\overset{\text{Vor}}{\le}} L$

Also

$$\underbrace{\Big[\ln U(s)\Big]_{s=a}^{s=t}}_{\ln U(t) - \ln C} = \int_a^t \frac{d}{ds} \ln U(s)\, ds \le \int_a^t L\, ds = L(t-a)$$

$$\Longrightarrow \quad \frac{U(t)}{\cancel{C}} \le C e^{L(t-e)} \quad , \quad \text{QED.}$$

Falls $C = 0$: $\quad$ Vor $\Rightarrow \quad u(t) \le \tilde{C} + L \int_a^t u(s)\, ds \quad \forall \tilde{C} > 0$

$$\overset{\text{oben}}{\Longrightarrow} \quad u(t) \le \tilde{C} e^{L(t-a)} \quad \forall \tilde{C} > 0$$

$$\Longrightarrow \quad u(t) \le 0 \text{ , tats. } u(t) = 0.$$

QED $\square$

Nochmal Matrix-Exponential

konkrete Berechnung von e^{At}:

Diagonalisieren.

Erinnerung: $\dot{\underline{x}} = A\underline{x} \Rightarrow \underline{x}(t) = e^{At}\underline{x}_0(0)$.

~~Es gilt~~

Falls A diag.bar, $A = SDS^{-1}$, $D = \mathrm{diag}(d_1 \dots d_n)$

Es gilt $$e^{SBS^{-1}} = \sum_{k=0}^{\infty} \frac{1}{k!} \left(SBS^{-1}\right)^k$$

$$(SBS^{-1})^2 = SB\cancel{S^{-1}S}BS^{-1} = SB^2S^{-1}$$

$$(SBS^{-1})^k = SB\cancel{(S^{-1}S)}B\cancel{(S^{-1}S)}\cdots\cancel{(S^{-1}S)}BS^{-1} = SB^kS^{-1}$$

$$\Rightarrow e^{SBS^{-1}} = \sum_{k=0}^{\infty} \frac{1}{k!} SB^kS^{-1}$$

$$= \lim_{K\to\infty} \sum_{k=0}^{K} \frac{1}{k!} S B^k S^{-1}$$

$$= \lim_{K\to\infty} S \left(\sum_{k=0}^{K} \frac{1}{k!} B^k \right) S^{-1}$$

$A \mapsto SAS^{-1}$ st.

$$= S \left(\lim_{K\to\infty} \sum_{k=0}^{K} \frac{1}{k!} B^k \right) S^{-1}$$

$$= \underline{\underline{S e^B S^{-1}}}$$

$$S = \begin{bmatrix} s_1 & \cdots & s_n \end{bmatrix}$$

$$\underline{A s_k} = S D S^{-1} s_k = S D e_k$$
$$= S d_k e_k = \underline{d_k s_k}$$

Anwendung auf $A = SDS^{-1}$:

$$e^{At} = e^{SDS^{-1}t} = S e^{Dt} S^{-1}$$

$$= S \begin{pmatrix} e^{d_1 t} & & \\ & \ddots & \\ & & e^{d_n t} \end{pmatrix} S^{-1}.$$

Anders gesagt:

Stelle Vektoren $\underline{x} \in \mathbb{R}^n$ in einer Eigenbasis von A dar (Spalten von S): $\underline{x} = S\underline{y}$

$$\Rightarrow \quad S\dot{\underline{y}} = \dot{\underline{x}} = A\underline{x} = AS\underline{y}$$

$$\Rightarrow \quad \dot{\underline{y}} = S^{-1}AS\underline{y} = \cancel{S^{-1}}S D \cancel{S^{-1}} S\underline{y}$$

$$\dot{\underline{y}} = D\underline{y}, \quad \underline{\dot{y}_j = d_j y_j}. \Rightarrow y_j(t) = y_{0,j}\, e^{d_j t}$$

$$= e^{d_j t} y_{0,j}$$

$$\Rightarrow \underline{y}(t) = e^{Dt}\underline{y}_0$$

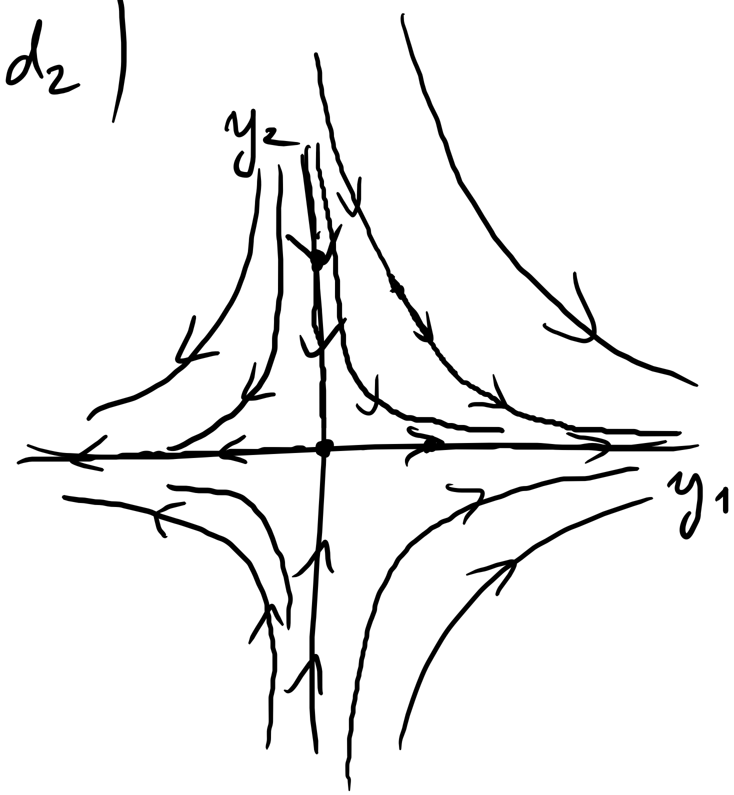
$$\Rightarrow \underline{x}(t) = S\underline{y}(t) = S e^{Dt} S^{-1} \underbrace{S\underline{y}_0}_{\underline{x}_0}.$$

<u>Bsp</u> $n=2$, $D = \begin{pmatrix} d_1 & \\ & d_2 \end{pmatrix}$

$d_1 > 0$, $d_2 < 0$

$y_1(t) = e^{d_1 t} y_{01}$

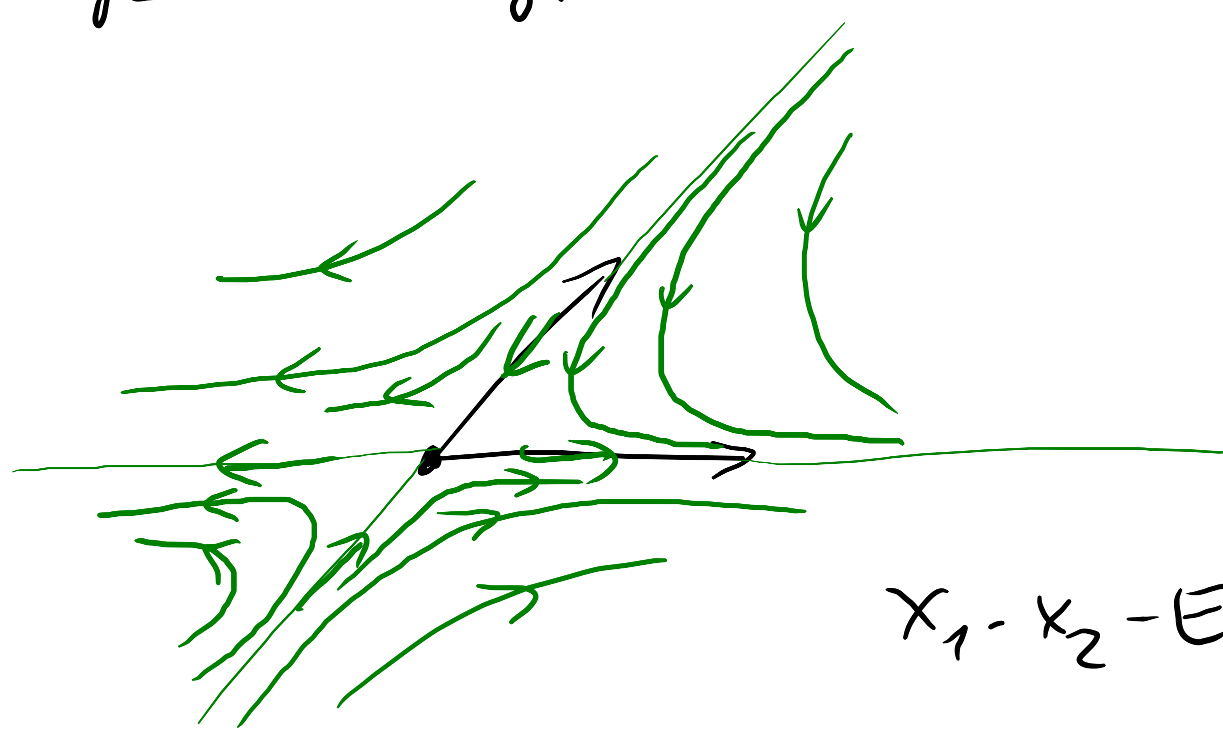
$y_2(t) = e^{d_2 t} y_{02}$



$$y_2(t) = (\text{const.})\, y_1(t)^{\frac{d_2}{d_1}}$$

$y_1(t) > 0$ $\qquad$ $y_2 = (\text{const.})\, |y_1|^{d_2/d_1}$

$\dot{\underline{x}} = A\underline{x}$

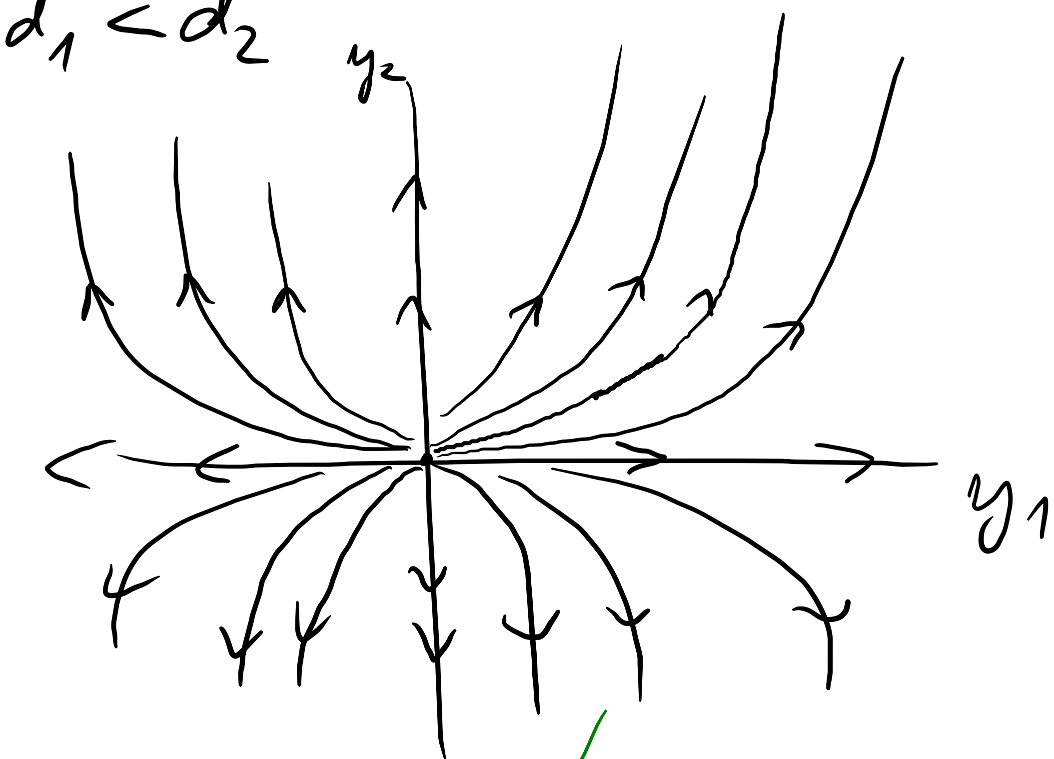


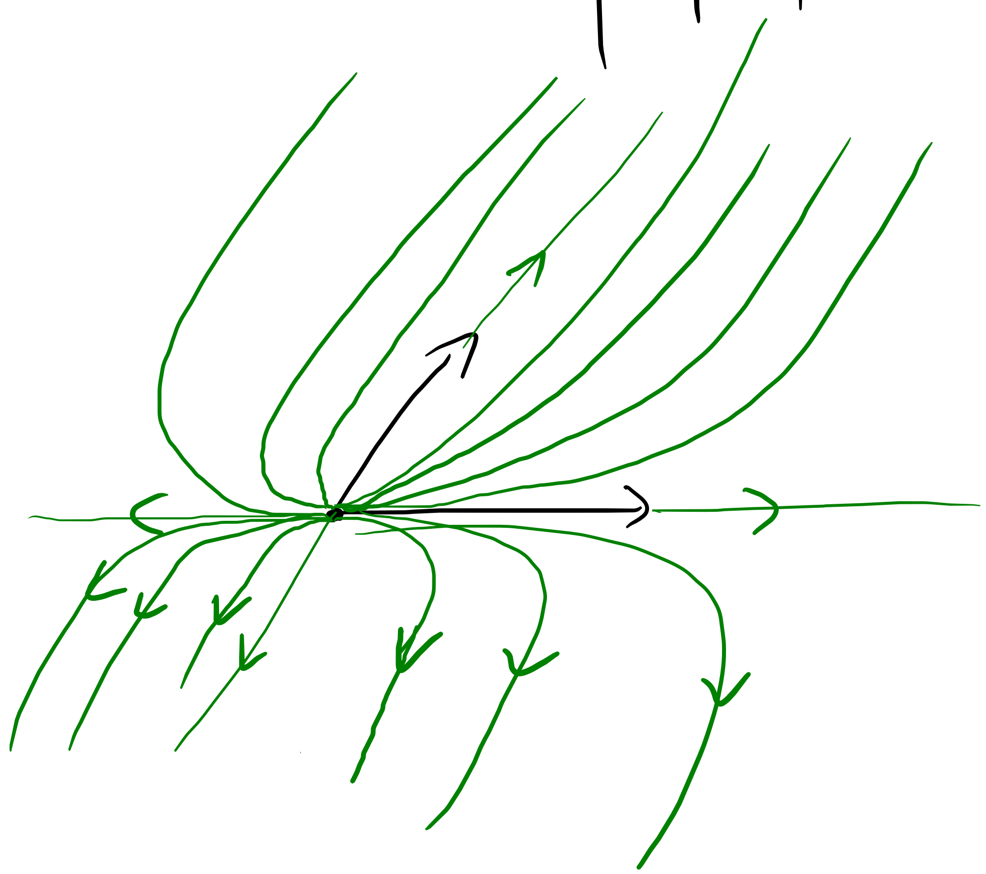
x_1-x_2-Ebene.

<u>Bsp</u> $n=2$, $0<d_1<d_2$

$$y_2(t) = (\text{const.})\, y_1(t)^{d_2/d_1}$$

$$y_2 = (\text{const.})\, y_1^{d_2/d_1}$$

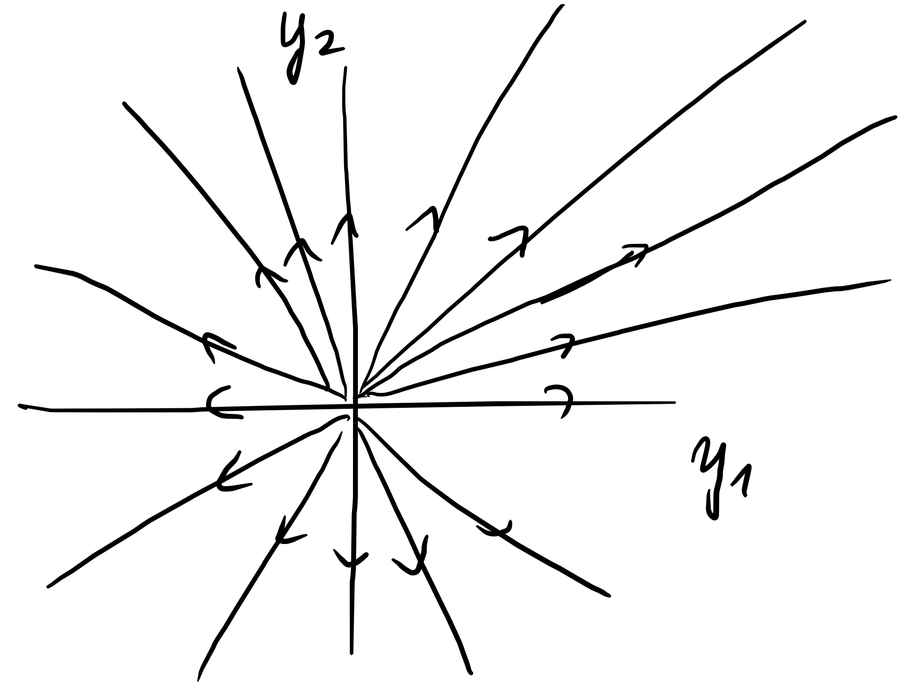




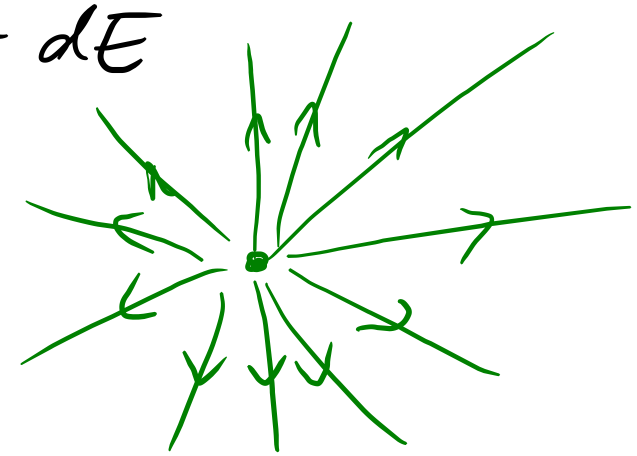
Bsp $n=2, \quad 0 < d_1 = d_2$

$$y_1(t) = e^{d_1 t}\, y_{01}$$

$$y_2(t) = e^{d_1 t}\, y_{02}$$



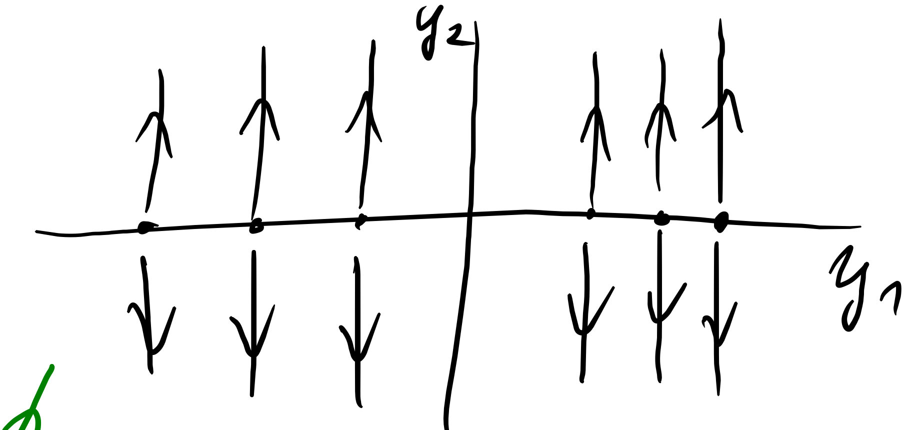
$$A = S \underbrace{D}_{dE} S^{-1} = d \; S \cancel{E} S^{-1} = dE$$

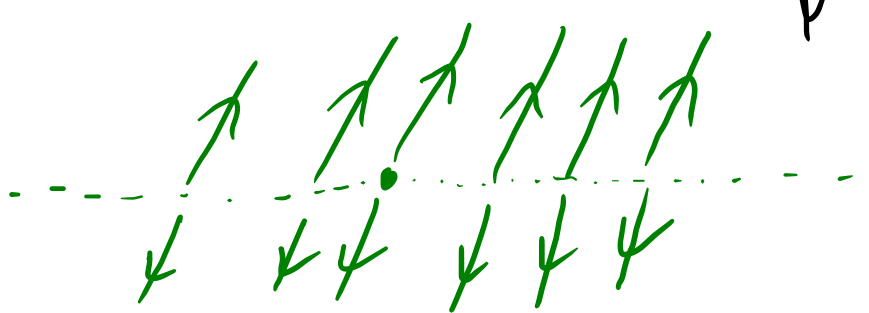


Bsp $n=2, \quad 0 = d_1 < d_2$

$$y_1(t) = y_{01}$$

$$y_2(t) = e^{d_2 t}\, y_{02}$$

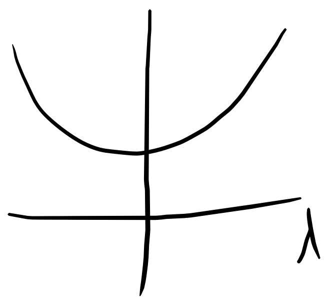




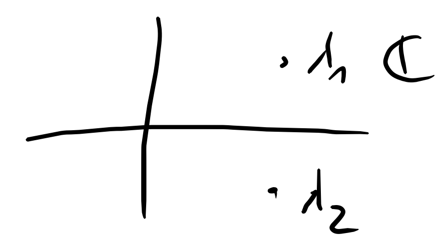
Falls A komplex diagbar.

z.B. $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$$P_A(\lambda) = \det \underbrace{\begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix}}_{A-\lambda E} = \lambda^2 + 1 = \lambda^2 - i^2$$
$$= (\lambda+i)(\lambda-i)$$



Wenn $A \in M_n(\mathbb{R})$, dann hat $P_A(\lambda)$ reelle Koeff., und NST entweder reell oder in konjugierten Paaren, $\lambda_1 \in \mathbb{C}$ und $\lambda_2 = \overline{\lambda_1} \in \mathbb{C}$



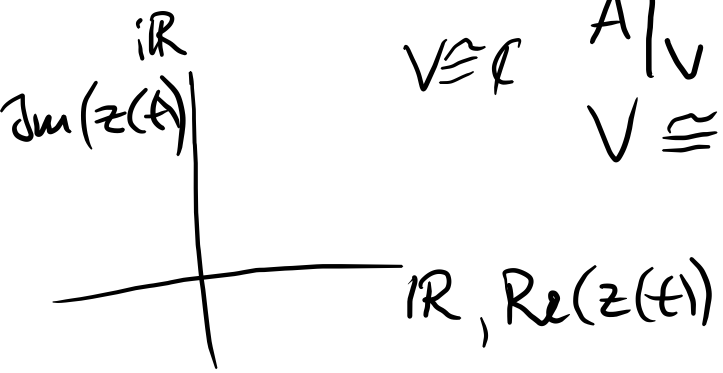
<u>Bew</u> $P_A(\overline{\lambda}) = \overline{P_A(\lambda)}$. $\square$

<u>Bsp</u> in $\mathbb{C}^n$ Eigenraum V, $\dim_{\mathbb{C}} V = 1$,

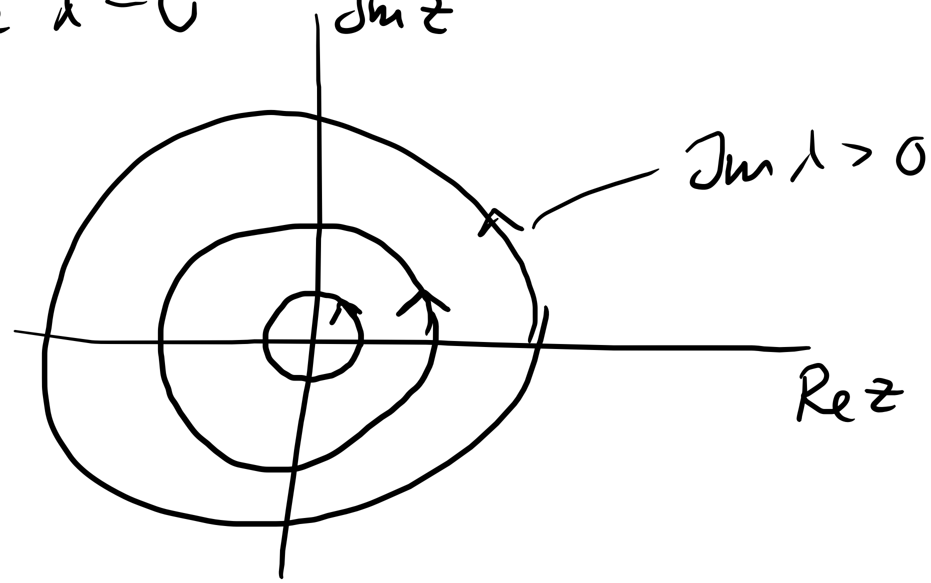
$V \cong \mathbb{C}$ $A|_V = \lambda$. Wähle Basis $\underline{v} \in V$,

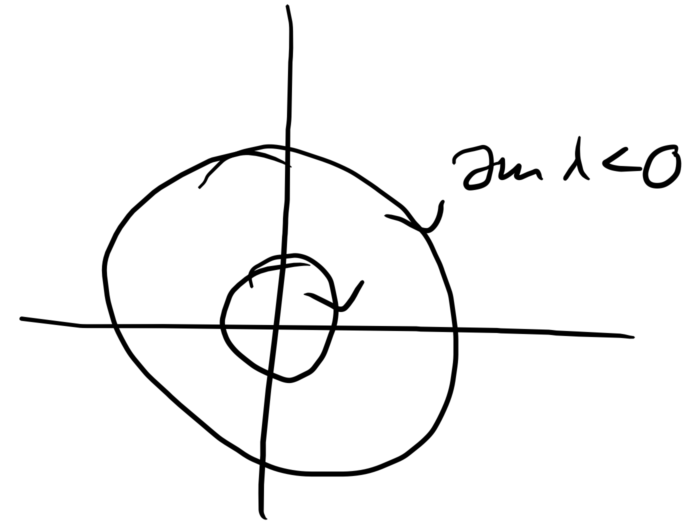
$V \cong \mathbb{C}$, $z(t) = e^{\lambda t} z_0 = e^{(Re(\lambda) + i\,Im(\lambda))t} z_0$

$$= e^{Re(\lambda)t}\, e^{i\,Im(\lambda)t}\, z_0$$

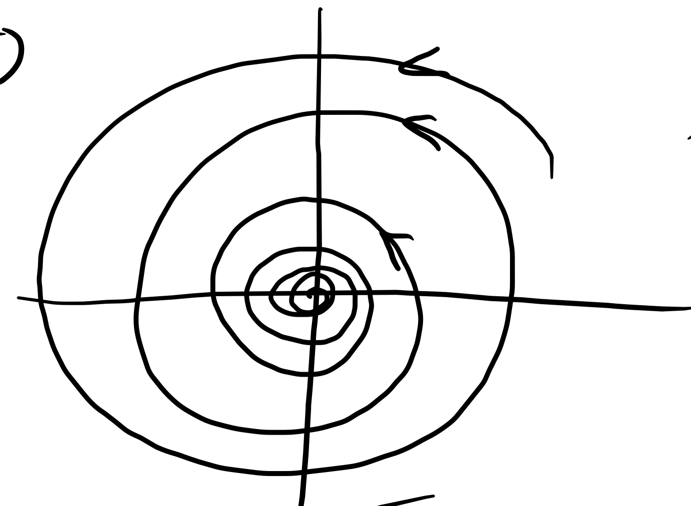


Bsp $\mathrm{Re}\,\lambda = 0$





Bsp $\mathrm{Re}\,\lambda < 0$

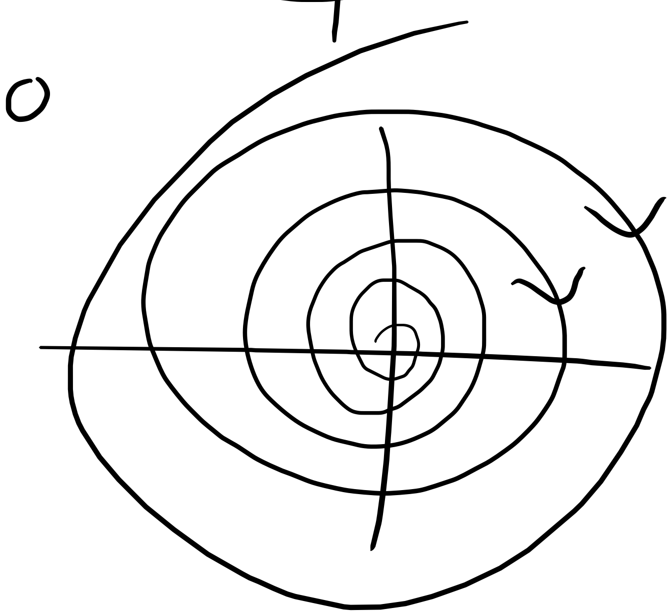


exp. Spirale

$$r = (\text{const.})\, e^{a\varphi}$$

$$a = \mathrm{Re}\,\lambda < 0, \quad z(t) \xrightarrow{t\to\infty} 0$$

Bsp $\mathrm{Re}\,\lambda > 0$



reelle Lsgen

Eigenraum V_1 mit EW λ_1 $\qquad \dim_{\mathbb{C}} V_1 = 1$

V_2 mit EW $\lambda_2 = \overline{\lambda_1}$ $\qquad \dim_{\mathbb{C}} V_2 = 1$

Wähle Basis $\underline{v}_1 \in V_1$, $\underline{v}_2 \in V_2$

$\Rightarrow \{\underline{v}_1, \underline{v}_2\}$ Basis in $V_1 + V_2$

$\underline{z}(t) = z_1(t)\,\underline{v}_1 + z_2(t)\,\underline{v}_2$, $\quad z_i(t) = e^{\lambda_i t} z_{0i}$

$$\Rightarrow \underline{z}(t) = \sum_{i=1}^{2} e^{\lambda_i t} z_{0i} \underline{v}_i \in \mathbb{C}^n$$

<u>Beh</u> $\underline{z}(t) \in \mathbb{R}^n \; \forall t \iff z_{02} \underline{v}_2 = \overline{z_{01} \underline{v}_1}$

<u>Bew</u> "$\Leftarrow$": wegen $\lambda_2 = \overline{\lambda_1}$ und $z + \overline{z} \in \mathbb{R} \; \forall z \in \mathbb{C}$.

"$\Rightarrow$": $t = 0 \Rightarrow \operatorname{Im}(z_{02} \underline{v}_2) = -\operatorname{Im}(z_{01} \underline{v}_1)$

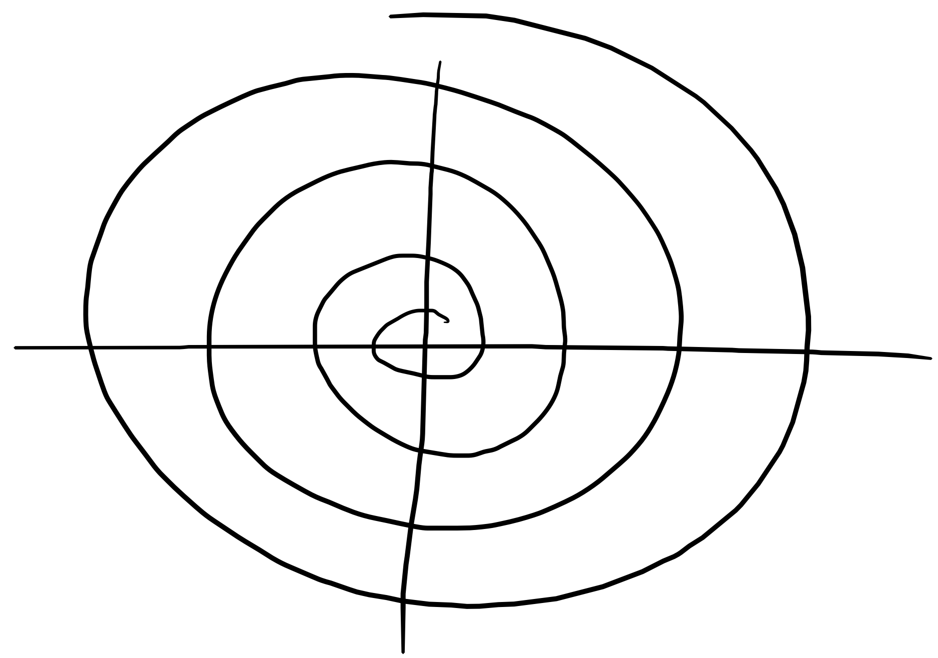
$t = \dfrac{\pi}{2 \operatorname{Im} \lambda_1} \Rightarrow \operatorname{Im}(i z_{02} \underline{v}_2) = -\operatorname{Im}(-i z_{01} \underline{v}_1)$

$\Rightarrow \operatorname{Re}(z_{02} \underline{v}_2) = \operatorname{Re}(z_{01} \underline{v}_1)$. $\square$

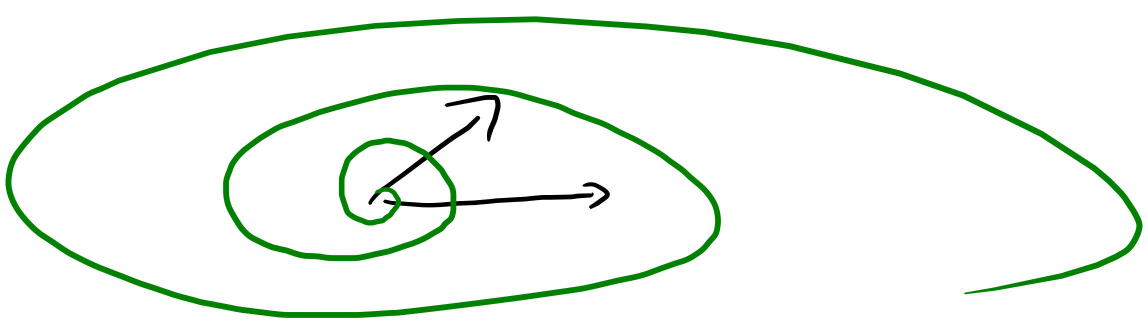
Dann $\underline{z}(t) = 2\,\mathrm{Re}\left(e^{\lambda_1 t} z_{01} \underline{v}_1\right)$

$$= \overbrace{2\, e^{\mathrm{Re}\,\lambda_1 t} \cos(\mathrm{Im}(\lambda_1) t)}^{y_1(t)}\, \mathrm{Re}\left(z_{01} \underline{v}_1\right)$$

$$+ \underbrace{2\, e^{\mathrm{Re}\,\lambda_1 t} \sin(\mathrm{Im}(\lambda_1) t)}_{y_2(t)}\, \underbrace{\mathrm{Re}\left(i z_{01} \underline{v}_1\right)}_{-\mathrm{Im}(z_{01} \underline{v}_1)}$$



Im $\mathbb{R}^q$!



Frage: Falls A nicht komplex diagbar?

$\longrightarrow$ Jordan-Normalform.