A8 a) $\ddot{x} + 4\dot{x} + 4x = 0$

Exp-Ansatz $x(t) = e^{\lambda t}$

$\Rightarrow \ddot{x} = \lambda^2 e^{\lambda t}, \quad \dot{x} = \lambda e^{\lambda t}$

$\Rightarrow \underbrace{\lambda^2 + 4\lambda + 4}_{(\lambda+2)^2} = 0$

$(\lambda+2)^2$, d.h. $\lambda = -2$ ist doppelte NST

$\Rightarrow x(t) = e^{-2t}$ ist Lösung

$x(t) = t e^{-2t}$ ist Lösung

Hier: $x(t) = c_1 e^{-2t} + c_2 t e^{-2t}$ ist allg. Lsg.

Alternativ-Lsg: Red. d. Ordnung $\underline{\dot{x}} = \underbrace{\begin{pmatrix} 0 & 1 \\ -4 & -4 \end{pmatrix}}_{A} \underline{x}$

Lsg $\underline{x}(t) = e^{At} \underline{x}_0$

Noch eine Vor. d. Konst $x(t) = c(t) e^{-2t} \Rightarrow \ddot{c} = 0$

A8 b) $\ddot{x} + 4\dot{x} + 4x = t^{-2} e^{-2t}$ $\quad (t > 0)$

Var. der Konst.: $x(t) = c_1(t) e^{-2t} + c_2(t)\, t\, e^{-2t}$

$\Rightarrow \dot{x} = \dot{c}_1 e^{-2t} - 2 c_1 e^{-2t} + \dot{c}_2 t e^{-2t} + c_2 e^{-2t} - 2 c_2 t e^{-2t}$

$\Rightarrow \ddot{x} = \dots$

$\Rightarrow \ddot{x} + 4\dot{x} + x = \cancel{\dot{c}_2 e^{-2t}} + \ddot{c}_1 e^{-2t} + \ddot{c}_2 e^{-2t} + 2\dot{c}_2 e^{-2t}$

$\overset{!}{=} t^{-2} e^{-2t}$

Brauchen nur 1 spez. ~~inho~~ Lsg des inhom. Problems:

z.B. $c_2 = 0$

$\ddot{c}_1 = t^{-2} \Rightarrow \dot{c}_1 = -t^{-1} + C$

$C := 0$

$\Rightarrow c_1 = -\ln t$

$\Rightarrow$ allg. Lsg $x(t) = \underline{\underline{-\ln(t) e^{-2t} + c_1 e^{-2t} + c_2 t e^{-2t}}}$.

Nochmal zu 8e) Red. d. Ordg;

$$x_1 = x \qquad \ddot{x} + 4\dot{x} + 4x = 0$$

$$x_2 = \dot{x}$$

$$\left\{\begin{array}{l} \dot{x}_1 = x_2 \\ \dot{x}_2 = -4x_2 - 4x_1 \end{array}\right\} \Leftrightarrow \underline{\dot{x}} = \begin{pmatrix} 0 & 1 \\ -4 & -4 \end{pmatrix} \underline{x}$$

<u>A5</u> $f(x,y) = \begin{cases} \frac{x^4+y^4}{x^2+y^2} & \text{für } (x,y) \neq (0,0) \\ 0 & \text{für } (x,y) = (0,0) \end{cases}$

partielle: $f(x,0) = x^2$ ist diffbar mit Abl. 0 in $x=0$

$$\Rightarrow \exists\, \partial_x f = 0.$$

$f(0,y) = y^2$ . . . . $\Rightarrow \exists\, \partial_y f = 0.$

total differbar $\overset{\text{Def}}{\Longleftrightarrow}$ $f(h_x, h_y) = \overbrace{f(0,0)}^{0} + \overbrace{\nabla f}^{0} \cdot \begin{pmatrix} h_x \\ h_y \end{pmatrix} + o(\|h\|)$

hier tot. diff. $\Longleftrightarrow$ $f(h_x, h_y) = o(\|h\|)$

$\Longleftrightarrow \frac{f(h)}{\|h\|} \xrightarrow{h \to 0} 0$

Polarkoo: $\left.\begin{aligned} x &= r\cos\varphi \\ y &= r\sin\varphi \end{aligned}\right\} \Rightarrow f(x,y) = \frac{r^4\cos^4\varphi + r^4\sin^4\varphi}{r^2}$

$= r^2 \underbrace{(\cos^4\varphi + \sin^4\varphi)}_{0 \leq \quad \leq 2} \leq 2r^2$

$h = (x,y)$
$\|h\| = r$

$\Rightarrow \frac{f(x,y)}{r} = r(\ldots) \leq 2r \xrightarrow{r \to 0} 0$

also ja.

Alternativ: $x^4 + y^4 \leq x^4 + 4x^2y^2 + y^4 = (x^2+y^2)^2 \Rightarrow f(x,y) \leq x^2+y^2 = o\left(\left\|\begin{pmatrix} x \\ y \end{pmatrix}\right\|\right)$

Alternativ: zeige, dann $\partial_x f$ und $\partial_y f$ st. sind.

A11 $\gamma : [0, 2\pi] \to \mathbb{R}^3$, $\gamma(t) = \begin{pmatrix} \sin 2t \\ \cos 3t \\ t^2 \end{pmatrix}$

$v : \mathbb{R}^3 \to \mathbb{R}^3$, $v\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2 x_3 \\ x_1 x_3 \\ x_1 x_2 \end{pmatrix}$

$$\int_\gamma \underline{v}(\underline{x}) \cdot d\underline{x} = ?$$

Lösungsweg: $\displaystyle\int_0^{2\pi} dt \begin{pmatrix} \cos(3t)\, t^2 \\ \sin(2t)\, t^2 \\ \sin(2t) \cos(3t) \end{pmatrix} \cdot \begin{pmatrix} 2\cos 2t \\ -3 \sin 3t \\ 2t \end{pmatrix} = \ldots$

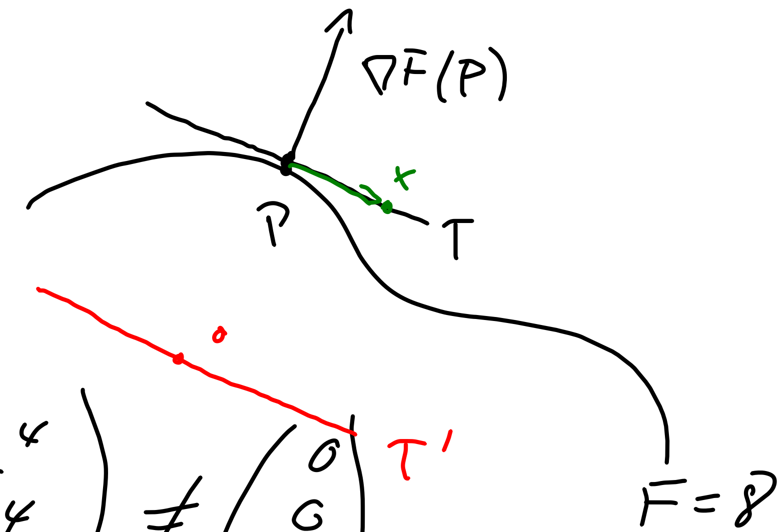
Alternativ-Lsg: $v = \nabla F$, $F(x_1, x_2, x_3) = x_1 x_2 x_3$

$$\int_\gamma \nabla F \cdot d\underline{x} = F(\gamma(2\pi)) - F(\gamma(0)) = \sin(4\pi)\cos(6\pi)(2\pi)^2 - \sin(0)\cos(0)\,0^2 = 0.$$

A6 3.3: Tangentialebene T an $x^5+y^5+z^5=8$

in $P=(a,b,c)$

ist $a^4x+b^4y+c^4z=8$.



$F(x,y,z)=x^5+y^5+z^5$.

$\nabla F(P)\perp T$ vorausgesetzt, dass $\nabla F(P)\neq 0$.

$$\nabla F = \begin{pmatrix}5x^4\\5y^4\\5z^4\end{pmatrix}, \quad \nabla F(P) = \begin{pmatrix}5a^4\\5b^4\\5c^4\end{pmatrix} \neq \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$T' = \{\underline{x} \mid \underline{x}\cdot\nabla F(P) < 0\}$$

$$T = \{\underline{x} \mid (\underline{x}-P)\cdot\nabla F(P) = 0\}$$

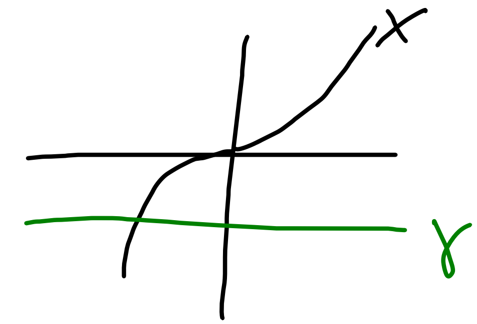
$$0 = \frac{1}{5}\left(\begin{pmatrix}x\\y\\z\end{pmatrix}-\begin{pmatrix}a\\b\\c\end{pmatrix}\right)\cdot\begin{pmatrix}5a^4\\5b^4\\5c^4\end{pmatrix} = \cancel{5}\begin{pmatrix}x\\y\\z\end{pmatrix}\cdot\begin{pmatrix}a^4\\b^4\\c^4\end{pmatrix} - \cancel{5}\left(\underbrace{a^5+b^5+c^5}_{=8=F(P)}\right)$$

$$\iff 8 = a^4x+b^4y+c^4z.$$

A 13 $\quad x^3 + \alpha x^2 + \beta x + \gamma = 0$

$g(\alpha, \beta, \gamma)$, z.B. $g(0,0,1) = -1$

d.h. $(-1)^3 + 0\cdot(-1)^2 + 0\cdot(-1) + 1 = 0$



Z.z. g C^1 in Umg von $(0,0,1)$

$Dg(0,0,1) = \left(-\frac{1}{3}, \frac{1}{3}, -\frac{1}{3}\right)$,

Satz über impl. Fkt. $\quad F(\alpha,\beta,\gamma,x) = x^3 + \alpha x^2 + \beta x + \gamma$

$\mathbb{R}^3 \times \mathbb{R}$

$F(\alpha,\beta,\gamma, g(\alpha,\beta,\gamma)) = 0$, $\quad g =$ impl. Fkt.

$F(0,0,1,-1) = 0$.

Vor: $\exists Y^{-1}$, $Y = D_y F \in M_n(\mathbb{R})$, hier heißt y x, $n = 1$.

brauchen $\frac{\partial F}{\partial x} \neq 0$, $\quad \frac{\partial F}{\partial x} = 3x^2 + 2\alpha x + \beta$, $\quad \frac{\partial F}{\partial x}(0,0,1,-1) = 3(-1)^2 = 3 \neq 0$.

SIF $\Rightarrow$ $\exists g$ in Umg. von $(0,0,1)$
$g \in C^1$

und $\quad F(\alpha,\beta,\gamma,g(\alpha,\beta,\gamma)) = 0$

∂_α : $\quad \frac{\partial F}{\partial \alpha} + \frac{\partial F}{\partial x}\frac{\partial g}{\partial \alpha} = 0 \Rightarrow \frac{\partial g}{\partial \alpha} = -\left(\frac{\partial F}{\partial x}\right)^{-1}\frac{\partial F}{\partial \alpha}$

hier: $\quad \frac{\partial F}{\partial \alpha} = x^2, \quad \frac{\partial F}{\partial \alpha}(0,0,1,-1) = 1$

$\frac{\partial F}{\partial x} = 3x^2 + 2\alpha x + \beta, \quad \frac{\partial F}{\partial x}(0,0,1,-1) = 3(-1)^2 = 3.$

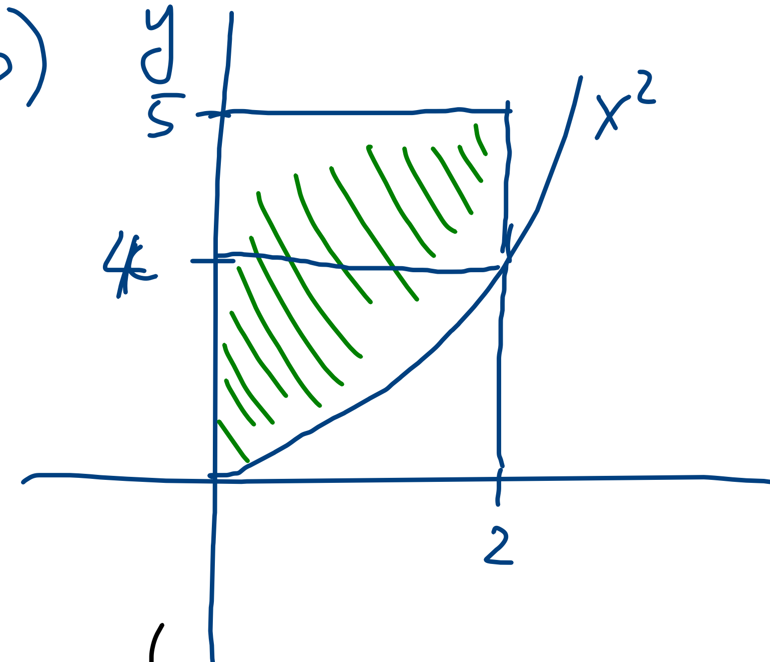
$\Rightarrow \quad \frac{\partial g}{\partial \alpha}(0,0,1) = -\frac{1}{3}\cdot 1 = -\frac{1}{3}.$

ebenso für $\frac{\partial g}{\partial \beta}, \frac{\partial g}{\partial \gamma}$.

A 9

$$I = \int_0^2 \left(\int_{x^2}^5 xy \, dy \right) dx$$

b)



c)

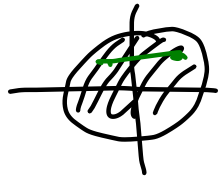
$$\int_{?}^{?} \left(\int_{?}^{?} xy \, dx \right) dy$$

Lsg 1:

$$\int_0^4 \left(\int_0^{\sqrt{y}} xy \, dx \right) dy + \int_4^5 \left(\int_0^2 xy \, dx \right) dy$$

Lsg 2:

$$\int_0^5 \left(\int_0^{\min(\sqrt{y},2)} xy \, dx \right) dy$$

A 14 $S :=$ 

$f: S \to \mathbb{R}$ C^1

Beh f Lipschitz

$\|f(x) - f(y)\| \leq L \|x-y\|$

~~$\|x-y\| \leq L \|f(x) - f(y)\|$~~

Wissen ∇f st., S kompakt $\Rightarrow$ $\sup_{x \in S} \|\nabla f(x)\| < \infty$

S konvex: $\gamma(t) = tx + (1-t)y$

$$\gamma : [0,1] \to S$$

$$|f(x) - f(y)| = \left| \int_\gamma \nabla f(\underline{u}) \cdot d\underline{u} \right|$$

$$\leq \int_0^1 dt \left| \left\langle \nabla f(\gamma(t)), \underbrace{\dot{\gamma}(t)}_{x-y} \right\rangle \right|$$

Cauchy-Schwarz $\leq \int_0^1 dt \, \|\nabla f(\gamma(t))\| \, \|x-y\|$

$$\leq \int_0^1 dt \quad \underbrace{\sup_{x \in S} \|Dg(x)\|}_{=:L} \quad \|x-y\|$$

$$= L \, \|x-y\| \qquad \square$$