Universität Tübingen, Fachbereich Mathematik
Dr. Stefan Keppeler Wintersemester 22/23

Groups and Representations

Material for our Zoom meeting on 22 December 2022

6.4 Lie Algebras (cont.)

Let us consider special curves though $e \in G$, namely one-parameter subgroups, i.e. solutions of

$$\dot{g}(t) = \mathrm{i}X g(t)\,, \quad g(0) = e\,,$$

with $X \in \mathfrak{g}$. We write $g(t) = \exp(\mathrm{i}Xt)$. For matrix Lie groups this exponential is given by the absolutely and uniformly convergent series (cf. Problem 26)

$$\exp(\mathrm{i}tX) = \sum_{\nu=0}^{\infty} \frac{(\mathrm{i}t)^\nu}{\nu!} X^\nu$$

For the special groups with $\det g = 1$ the generators are traceless, since

$$\det g(t) = \det(\mathrm{e}^{\mathrm{i}tX}) = \mathrm{e}^{\mathrm{i}t\,\mathrm{tr}\,X} \stackrel{!}{=} 1 \qquad \Leftrightarrow \qquad \mathrm{tr}\,X = 0\,.$$

For unitary groups, i.e. $gg^\dagger = \mathbb{1}$, the generators are Hermitian, since

$$g(t)^\dagger = g(t)^{-1} \quad \Leftrightarrow \quad \mathrm{e}^{-\mathrm{i}tX^\dagger} = \mathrm{e}^{-\mathrm{i}tX} \quad \Leftrightarrow \quad X = X^\dagger\,.$$

Examples:

1. SO(3), defining rep, Lie algebra, structure constants:

   https://youtu.be/5ud3zMg-epo (5 min) (1)

2. O_A operators for SO(3):

   https://youtu.be/Ia7SX4PrmNo (7 min) (2)

6.5 More on SO(3)

Parametrise rotations as $R_{\vec{n}}(\psi)$, with rotation angle ψ and rotation axis $\vec{n}$:

https://youtu.be/VBitdDXs9XQ (4 min) (3)

Topology of SO(3):

https://youtu.be/bUu6amDkNb0 (3 min) (4)

Further observation: Rotations about a fixed axis form a (one-parameter) subgroup of SO(3). Such a subgroup is isomorphic to SO(2) (cf. Section 6.2). For arbitrary rotations $R \in \mathrm{SO}(3)$ we have (by explicit calculation using the generators from above)

$$R R_{\vec{n}}(\psi) R^{-1} = R_{\vec{n}'}(\psi) \qquad \text{with} \quad \vec{n}' = R\vec{n}\,.$$

This implies that all rotations by the same angle are in the same conjugacy class.