Universität Tübingen, Fachbereich Mathematik — Wintersemester 22/23
Dr. Stefan Keppeler & Saradha Senthil-Velu

Groups and Representations

Homework Assignment 4 (due on 16 November 2021)

Problem 13
Let $(\mathbb{R},+)$ be the additive group of real numbers, and

$$\Gamma(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad x \in \mathbb{R},$$

a representation on $\mathbb{C}^2$. Find all invariant subspaces. Is Γ completely reducible?

Problem 14
Let G be a finite group and $\Gamma : G \to \mathrm{GL}(V)$ a finite dimensional representation. Prove that $|\det \Gamma(g)| = 1 \; \forall \, g \in G$.

Problem 15
Let G be a finite group, $|G| = n$. We number the group elements, $G = \{g_j, j = 1 \ldots n\}$, denote by m the number of conjugacy classes c (with n_c elements) and by p the number of non-equivalent irreducible representations Γ^i of G (with dimensions d_i).

Consider the matrix U with entries $u_{ja} = \sqrt{\frac{d_{i_a}}{n}}\,\Gamma^{i_a}(g_j)_{\mu_a\nu_a}$ with a triple $a = (i_a, \mu_a, \nu_a)$.

Employ the results of Sections 2.5 and 2.6 in order to solve the following problems.

a) Determine the dimensions of U and express the orthogonality relation for irreducible representations (Theorem 6) in terms of U.

b) Show:

(i) $\sum_{i \le p} d_i \operatorname{tr}\left(\Gamma^i(g_j)\Gamma^i(g_k)^\dagger\right) = n\delta_{jk}$,

(ii) $\sum_{g \in c} d_i \, \Gamma^i(g) = n_c \, \chi^i_c \, \mathbb{1}$ and

(iii) $\sum_{i \le p} n_c \, \chi^i_c \, \overline{\chi^i_{c'}} = n\delta_{cc'}$.

c) Conclude that $m = p$.

Problem 16 (Continuation of Problem 12)
We now determine all irreducible representations of D_4 (up to equivalence):

d) What are the dimensions of the irreducible representations?

e) Find all one dimensional irreducible representations.
Hint: First consider irreducible representations of quotient groups, cf. the remarks on (non-)faithful representations in Section 2.1.

f) Determine the character table and the remaining representation(s).

Problem 17

Let V be a finite-dimensional vector space and $P : V \to V$ a linear operator with $P^2 = P$.

a) Show that there exist subspaces U and W with $V = U \oplus W$, $P|_U = \mathbb{1}$ and $P|_W = 0$.

Let $\langle \cdot, \cdot \rangle$ be a scalar product on V, and let $P^\dagger = P$.

b) Show that $U = W^\perp$.