Universität Tübingen, Fachbereich Mathematik
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Groups and Representations

Homework Assignment 7 (due on 7 December 2022)

Problem 24
$V = \mathbb{C}^2$ carries the 2-dimensional irreducible representation of $D_3 \cong S_3$ (cf. Section 2.4.1). On $W = V \otimes V$ we consider the corresponding product representation. Decompose W into irreducible invariant subspaces by applying the generalised projection operators, and determine the Clebsch-Gordan coefficients.

Problem 25
Let V be a vector space and $A : V \to V$ a linear map. Show that if A is nilpotent (i.e. if for some $n \in \mathbb{N}$ we have $A^n v = 0 \; \forall \, v \in V$) then $\operatorname{tr} A = 0$.

Problem 26
For $A \in \mathbb{C}^{n\times n}$ the matrix exponential is defined as

$$\mathrm{e}^A = \exp(A) = \sum_{\nu=0}^{\infty} \frac{A^\nu}{\nu!} \,.$$

Prove:

a) The series converges absolutely and uniformly.
HINT: On $\mathbb{C}^{n\times n}$ use the operator norm

$$\|A\| = \sup_{v \in \mathbb{C}^n \setminus \{0\}} \frac{|Av|}{|v|} \,,$$

for which we have $\|AB\| \leq \|A\| \, \|B\|$.

b) For $T \in \mathrm{GL}(n)$ we have $\mathrm{e}^{TAT^{-1}} = T\mathrm{e}^A T^{-1}$.

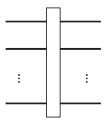
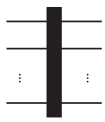
c) e^{tA} is the unique solution of the initial value problem $\dot{X}(t) = AX(t)$, $X(0) = \mathbb{1}$.

d) For $t, s \in \mathbb{C}$ we have $\mathrm{e}^{(t+s)A} = \mathrm{e}^{tA}\mathrm{e}^{sA}$.

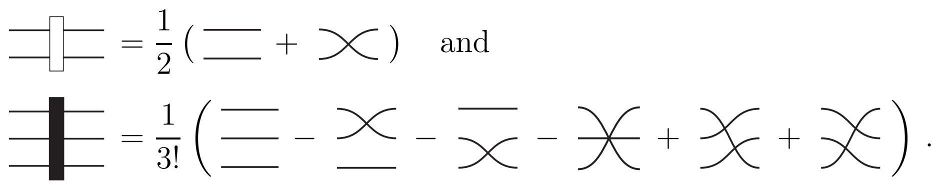
e) $\left(\mathrm{e}^A\right)^\dagger = \mathrm{e}^{(A^\dagger)}$.

f) $\det \mathrm{e}^A = \mathrm{e}^{\operatorname{tr} A}$.

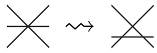
Problem 27
We can also write elements of the $\mathcal{A}(S_n)$ in birdtrack notation. In particular, we denote symmetrisers and anti-symmetrisers by open and solid bars, respectively, i.e.

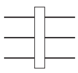
$\frac{1}{n!}s = \frac{1}{n!} \sum_{p \in S_n} p =$  and $\frac{1}{n!}a = \frac{1}{n!} \sum_{p \in S_n} \mathrm{sgn}(p) p =$  .

Note that we include a factor of $\frac{1}{n!}$ in the definition of bars over n lines. For instance,

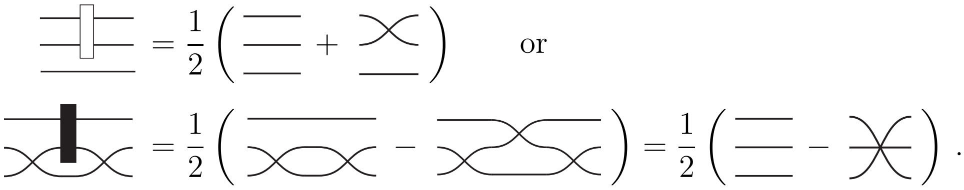


(∗)

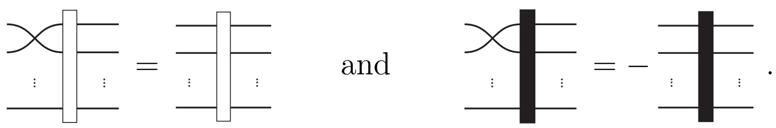
Notice that in birdtrack notation the sign of a permutation, $(-1)^K$, is determined by the number K of line crossings; if more than two lines cross in a point, one should slightly perturb the diagram before counting, e.g.  (K=3).

a) Expand  and  as in (∗).

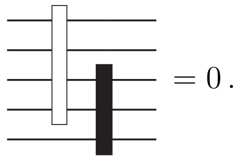
We also use the corresponding notation for partial (anti-)symmetrisation over a subset of lines, e.g.



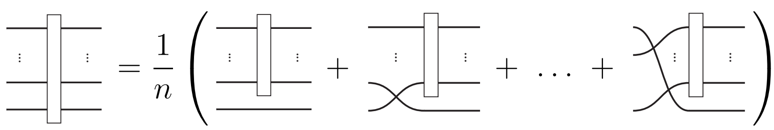
It follows directly from the definition of S and A that when intertwining any two lines S remains invariant and A changes by a factor of (-1), i.e.

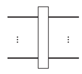


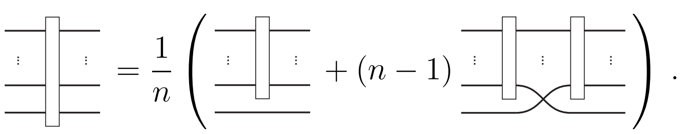
b) Explain why this implies that whenever two (or more) lines connect a symmetriser to an anti-symmetrizer the whole expression vanishes, e.g.



Symmetrisers and anti-symmetrisers can be built recursively. To this end notice that on the r.h.s. of



we have sorted the terms according to where the last line is mapped – to the nth, to the $(n-1)$th, . . . , to the first line line. Multiplying with  from the left and disentangling lines we obtain the compact relation



c) Derive the corresponding recursion relation for anti-symmetrisers.