Universität Tübingen, Fachbereich Mathematik
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Groups and Representations

Homework Assignment 8 (due on 14 December 2022)

Problem 28
We consider the abelian group $C_3 = \{e, a, a^{-1}\} \cong \mathbb{Z}_3$.

a) How many (non-equivalent) irreps does C_3 have, what are their dimensions and how often do they appear in the regular rep?

b) Show that
$$e_1 = \frac{1}{3}(e + a + a^{-1})$$
is a primitive idempotent, generating the trivial rep.

c) Use the ansatz
$$e_2 = xe + ya + za^{-1}$$
in order to find all primitive idempotents.

d) For each primitive idempotent find out whether it generates a new (non-equivalent) irrep or an irrep equivalent to one generated by a previous idempotent.

e) Specify all minimal left ideals and construct the corresponding irreps of C_3. Collect your results in a table.

Problem 29
Determine once more the characters of the irreps of S_3 by using the methods of Section 4.3.1.

Problem 30
Let V be a (complex, finite dimensional) vector space and let V^* be its dual, i.e. the space of all linear maps $V \to \mathbb{C}$. For a linear map $A : V \to V$ we define its dual $A^* : V^* \to V^*$ by $V^* \ni f \mapsto A^*(f) = f \circ A$. Let G be a group and $\Gamma : G \to \mathrm{GL}(V)$ a representation.

a) Define a representation $\Gamma^* : G \to \mathrm{GL}(V^*)$ in a natural way.
HINT: Simply replacing $\Gamma(g) : V \to V$ by its dual map doesn't quite work (why?) but with a slight modification it does.

Let $\{e_j\}$ be a basis of V and $\{f_j\}$ the corresponding dual basis, i.e. $f_j(e_k) = \delta_{jk}\ \forall\, j, k = 1, \ldots, \dim V = \dim V^*$. For $g \in G$ we express $\Gamma(g) : V \to V$ and $\Gamma^*(g) : V^* \to V^*$ as matrices in the bases $\{e_j\}$ and $\{f_j\}$, respectively.

b) What is the relation between these two matrices? What happens if Γ is unitary?

Problem 31

Let $\vec{n} \in S^2 \hookrightarrow \mathbb{R}^3$ be a unit vector in $\mathbb{R}^3$ and $\varphi \in \mathbb{R}$. We denote by σ_j, $j = 1, 2, 3$, the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} ,$$

and we define

$$\vec{\sigma} = \begin{pmatrix} \sigma_1 & \sigma_2 & \sigma_3 \end{pmatrix} .$$

Show that

$$\exp\left(-\mathrm{i}\tfrac{\varphi}{2}\vec{\sigma}\cdot\vec{n}\right) = \mathbb{1}\cos\tfrac{\varphi}{2} - \mathrm{i}\vec{\sigma}\cdot\vec{n}\sin\tfrac{\varphi}{2} ,$$

and verify that $\exp\left(-\mathrm{i}\tfrac{\varphi}{2}\vec{\sigma}\cdot\vec{n}\right) \in \mathrm{SU}(2)$.

HINT: First calculate $(\vec{\sigma}\cdot\vec{n})^2$.