Universität Tübingen, Fachbereich Mathematik
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Groups and Representations

Homework Assignment 10 (due on 11 January 2023)

Problem 36
The Lie algebra of SU(2) is the (real) vector space

$$\mathfrak{su}(2) = \{X \in \mathbb{C}^{2\times 2} : \operatorname{tr}(X) = 0, X^\dagger = X\}\,.$$

A basis is given by the Pauli matrices (see Problem 31). Show:

a) SU(2) acts on $\mathfrak{su}(2)$ by conjugation: $X \mapsto UXU^\dagger$.

b) $\langle X, Y\rangle = \frac{1}{2}\operatorname{tr}(XY)$ defines a scalar product on $\mathfrak{su}(2)$.
HINT: Begin by calculating $\operatorname{tr}(\sigma_i\sigma_j)$.

c) Every $U \in \mathrm{SU}(2) \cong S^3$ (cf. Problem 20) can be written as $\mathrm{e}^{-\frac{1}{2}\mathrm{i}\alpha\vec{\sigma}\cdot\vec{n}}$ with $\vec{n} \in S^2 \hookrightarrow \mathbb{R}^3$ (cf. Problem 31). Over which values does α run?

Problem 37
We define $\mathfrak{sl}(2,\mathbb{C}) := \{A \in \mathbb{C}^{2\times 2} : \operatorname{tr} A = 0\}$. Then the matrix exponential is a map

$$\exp : \mathfrak{sl}(2,\mathbb{C}) \to \mathrm{SL}(2,\mathbb{C}) = \{B \in \mathbb{C}^{2\times 2} : \det B = 1\}\,.$$

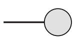
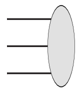
a) Show that the matrix

$$S_a = \begin{pmatrix} -1 & a \\ 0 & -1 \end{pmatrix}$$

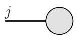
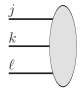
is in the image of exp iff $a = 0$.

b) Is $\mathrm{SL}(2,\mathbb{C})$ compact?

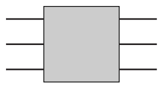
Problem 38
In birdtrack notation we write vectors $v \in V$ or tensors $t \in V^{\otimes n}$, as blobs with one or several lines attached, e.g.

$$v = $$  $$\in V \qquad \text{or} \qquad t =$$  $$\in V^{\otimes 3}.$$

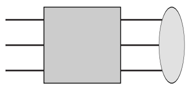
Expressed in a basis, components are

$$v_j = $$  $$\qquad \text{or} \qquad t_{jk\ell} =$$  $$,$$

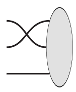
i.e. we write indices on the lines. Linear maps $A : V^{\otimes n} \to V^{\otimes n}$ are represented by blobs with n legs on each side, e.g.

$$A = $$  $$: V^{\otimes 3} \to V^{\otimes 3},$$

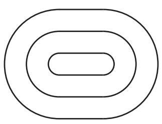
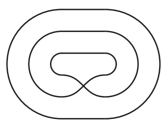
and with $t \in V^{\otimes 3}$ we have

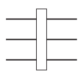
$$At = $$  $$\in V^{\otimes 3}.$$

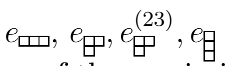
Birdtracks for permutations $p \in S_n$, or $\in \mathcal{A}(S_n)$, see Problems 23 & 27, can now be applied (as linear maps) to elements of $V^{\otimes n}$, e.g.

$$(12)t = $$  $$.$$

We obtain traces of linear maps by connecting the first line on the left to the first line on the right etc. (cf. also Problem 23), with each loop contributing a factor of $\dim V =: N$ (why?), e.g. for $e, (12) \in \mathcal{A}(S_3)$ we get

$$\operatorname{tr} e = $$  $$= N^3 \qquad \text{and} \qquad \operatorname{tr}(12) =$$  $$= N^2$$

a) Calculate the trace of $(132) \in S_3$ and the trace of  $\in \mathcal{A}(S_3)$.

b) Normalise the Young operators  $\in \mathcal{A}(S_3)$ of Section 5.3 s.t. they are idempotent. Determine the traces of these primitive idempotents. Later we will see that these are the dimensions of $\mathrm{GL}(N)$ irreps contained in $V^{\otimes 3}$.

HINT: Some identities from Problem 27 are useful.

Merry Christmas and Happy New Year!