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Groups and Representations

Homework Assignment 11 (due on 18 January 2023)

Problem 39
The elements of $\mathfrak{su}(2)$ can be written as $X = \vec{\sigma}\cdot\vec{x}$ with $\vec{x} \in \mathbb{R}^3$ (cf. Problems 31 & 36). The action of SU(2) on $\mathfrak{su}(2)$ by conjugation (see Problem 36) then defines a homomorphism

$$\begin{aligned} \varphi &: \mathrm{SU}(2) \to \mathrm{GL}(3,\mathbb{R}) \\ \vec{\sigma}\cdot\varphi(U)\vec{x} &:= U(\vec{\sigma}\cdot\vec{x})U^\dagger\,. \end{aligned}$$

Show that

a) $\varphi(U)_{ij} = \frac{1}{2}\operatorname{tr}(\sigma_i U \sigma_j U^\dagger)$,

b) $\varphi(U)^T = \varphi(U)^{-1}$, and

c) $\det(\varphi(U)) = 1$. HINT: Recall the connectedness properties of SU(2).

Hence $\varphi(\mathrm{SU}(2)) \subset \mathrm{SO}(3)$.

d) Determine the kernel of φ.

e) Calculate $\varphi(U_\alpha)$ for $U_\alpha = \mathrm{e}^{-\frac{1}{2}\mathrm{i}\alpha\sigma_3}$, $\alpha \in [0, 2\pi)$ and explain that $\varphi(\mathrm{SU}(2)) = \mathrm{SO}(3)$. What can we now conclude using the homomorphism theorem (Problem 8)?

Problem 40

a) Determine the Haar measure for SU(2) in axis-angle parametrisation,

$$U = \exp\left(-\mathrm{i}\tfrac{\alpha}{2}\vec{\sigma}\cdot\vec{x}\right)\,,$$

with $0 \le \alpha \le 2\pi$ and $\vec{x} \in S^2 \hookrightarrow \mathbb{R}^3$. Normalise s.t. $\mathrm{vol}(\mathrm{SU}(2)) = 1$.

HINT: It is convenient to first show $(\vec{x}\cdot\vec{\sigma})(\vec{y}\cdot\vec{\sigma}) = \mathbb{1}\vec{x}\cdot\vec{y} + \mathrm{i}\vec{\sigma}(\vec{x}\times\vec{y})$ and to use the unit vectors $\vec{e}_r, \vec{e}_\vartheta, \vec{e}_\varphi$ for spherical coordinates.

b) Use the result of (a) together with the results of Problem 39 in order to determine the Haar measure for SO(3) in the axis-angle parametrisation.