Universität Tübingen, Fachbereich Mathematik — Wintersemester 22/23
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# Groups and Representations

Homework Assignment 13 (due on 1 February 2023)

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**Problem 44** (continuation of Problem 43)
Let $K : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ be the Killing form from Problem 43, and let $G$ be such that $K$ is positive definite. We choose an orthonormal basis $\{X_j\}$ with respect to $K$, i.e. $K(X_j, X_k) = \delta_{jk}$, and define $C_2 \in E(\mathfrak{g})$ by

$$C_2 = \sum_j X_j X_j \, .$$

Show:

c) $C_2$ is independent of the choice of basis.

d) $C_2$ is a Casimir operator (the so-called *quadratic Casimir operator*), i.e.

$$\mathrm{Ad}_g(C_2) = C_2 \quad \forall\, g \in G \, .$$

**Problem 45**
Let $|j\rangle$, $j = 1, \ldots, N$, be the canonical basis for $V \cong \mathbb{C}^N$, and let $\mathrm{GL}(N)$ act on $V$ in the defining representation, i.e. $g|j\rangle = |k\rangle g_{kj}$ for $g \in \mathrm{GL}(N)$ (recall that we sum over repeated indices). For $V \otimes V$ choose the product basis $|jk\rangle = |j\rangle \otimes |k\rangle$, $j, k = 1, \ldots, N$, and let $\mathrm{GL}(N)$ act as follows,

$$g|jk\rangle = |j'k'\rangle g_{j'j}\, g_{k'k} \, .$$

Specialise to $N = 2$ and consider $e_{\substack{\square\\\square}} V \otimes V$. We obtain a basis for $e_{\substack{\square\\\square}} V \otimes V$ by applying $e_{\substack{\square\\\square}}$ to basis tensors $|jk\rangle$.

**Show** that $e_{\substack{\square\\\square}} V \otimes V$ is invariant under $\mathrm{GL}(N)$ by explicitly calculating the representation of $G$ carried by $e_{\substack{\square\\\square}} V \otimes V$. To this end apply $g \in \mathrm{GL}(2)$ to a basis of $e_{\substack{\square\\\square}} V \otimes V$ and read off the representation. Is this representation irreducible?

Repeat with $\mathrm{GL}(2)$ replaced by $\mathrm{SU}(2)$. Which representation do we obtain now?