Universität Tübingen, Fachbereich Mathematik
Dr. Stefan Keppeler Sommersemester 2021

Groups and Representations

Instruction 1 for the preparation of the lecture on 21 April 2021

1.2 Basic notions[1]

Definition: (group)
Let $G \neq \emptyset$ be a set and let $\circ$ be an operation $\circ : G \times G \to G$. We call $(G, \circ)$ a group if:

(G1) $a, b \in G \Rightarrow a \circ b \in G$ (closure) (already implied by $\circ : G \times G \to G$)

(G2) $(a \circ b) \circ c = a \circ (b \circ c) \; \forall \; a, b, c \in G$ (associativity)

(G3) $\exists \; e \in G$ with $a \circ e = a = e \circ a \; \forall \; a \in G$ (identity/neutral element)

(G4) for each $a \in G \; \exists \; a^{-1} \in G$ with $a \circ a^{-1} = e = a^{-1} \circ a$, with e from (G3) (inverses)

Definition: (abelian group)
A group $(G, \circ)$ is called commutative or abelian, if in addition we have:
(G5) $a \circ b = b \circ a \; \forall \; a, b \in G$ (commutativity)

Remarks:

1. The identity e is unique. For each $a \in G$ the corresponding inverse is unique.
   Can you show this?
2. We often call the operation multiplication and write $a \cdot b$ or just ab instead of $a \circ b$.
3. If the number of group elements is finite, we speak of a *finite group*, and we call the number of group elements the *order* $|G|$ of the group. (otherwise: *infinite group*).
4. A finite group is completely determined by its *group table*:

   https://youtu.be/gmTSAOSs9U0 (4 min) (1)

   No two elements within one row (or column) can be the same. (see exercises)
   This implies the *rearrangement lemma*: If we multiply all elements of a group $\{e, a, b, c, \ldots\}$ by one of the elements, we obtains again all elements, in general in a different order.

Examples:

trivial group, $(\mathbb{Z}, +)$, and $(\mathbb{R} \setminus \{0\}, \cdot)$ https://youtu.be/FuUWrnBVstQ (2 min) (2)

symmetry group of an object https://youtu.be/o1OM_fzkObA (4 min) (3)

Definition: (subgroup)
Let $(G, \circ)$ be a group. A subset $H \subseteq G$, which satisfies (G1)–(G4) (with the same operation $\circ$), is called a subgroup of G.

[1] section numbering according to lecture notes.

Remarks:

1. Every group has two trivial subgroups: $\{e\}$ and G.
   All other subgroups are called non-trivial.
2. If G is finite then $|H|$ divides $|G|$. (proof later)

Definition: (homomorphism)
Given two groups $(G, \circ)$ and $(G', \bullet)$, a map $f : G \to G'$ is called a homomorphism, if

$$f(a \circ b) = f(a) \bullet f(b) \quad \forall\, a, b \in G\,.$$

Remarks:

1. A homomorphism f maps the identity to the identity and inverses to inverses, more precisely $f(e_G) = e_{G'}$ and $f(a^{-1}) = f(a)^{-1}\ \forall\, a \in G$.
   Can you show this?
2. The *image* of the homomorphism $f : G \to G'$ is

$$\mathrm{im}(f) = f(G) = \{f(g)\ :\ g \in G\}\,,$$

   the *kernel* of f is the preimage of the identity of G' ,

$$\ker(f) = \{g \in G\ :\ f(g) = e_{G'}\}\,.$$

Definition: (isomorphism)
A bijective homomorphism $f : G \to G'$ is called isomorphism. We then say that G and G' are isomorphic, and write $G \cong G'$.

Remark: Isomorphic groups have the same group table, i.e. they are identical except for what we call their elements (and the group operation). (similarly for infinite groups)

1.3 Examples and outlook

Up to isomorphy there is only one group with two elements – but it comes in many guises:

https://youtu.be/7e8SMpY4Fk4 (9 min) (4)

Often we encounter cyclic groups:

https://youtu.be/TwITmOaX1gA (2 min) (5)

Functions that transform in a special way under a group will provide an interesting playing field for groups:

https://youtu.be/UdH9UIU5UCY (6 min) (6)

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Sommersemester 2021

Groups and Representations

Instruction 2 for the preparation of the lecture on 26 April 2021

1.4 Permutations – the symmetric group

Definition: (symmetric group)
The symmetric group of degree n, S_n, are the bijections of $\{1, 2, \dots, n\}$ to itself under composition.

Remarks:

1. Elements of S_n are called permutations.
2. $|S_n| = n!$

We use three notations for permutations:

two-line notation	`https://youtu.be/0mjbR0pjkFs` (1 min)	(1)
cycle notation	`https://youtu.be/kvISarU6UWA` (5 min)	(2)
birdtrack notation	`https://youtu.be/1hllM7IPf3M` (4 min)	(3)

Examples:

1. $S_2 = \{e, (12)\} \cong \mathbb{Z}_2$
2. $S_3 = \{e, (12), (13), (23), (123), (132)\}$
   Construct the group table! Is S_3 abelian?
   subgroups: $\{e\}$ and S_3 (trivial)
   $\{e, (12)\}$, $\{e, (13)\}$, $\{e, (23)\}$, all $\cong \mathbb{Z}_2$
   $\{e, (123), (321)\} \cong C_3$

Theorem 1. (Cayley)
Every group of order n is isomorphic to a subgroup of S_n.

Proof: `https://youtu.be/r4_oD2o6aqo` (5 min) (4)

Fun exercise (optional): Watch the video *An Impossible Bet* by `minutephysics`,

`https://youtu.be/eivGlBKlK6M` (2 min) (5)

and come up with a good strategy. Don't watch the solution! Think about cycles instead.

1.5 Group actions

Definition: (group action)
Let G be a group and M a set. A (group) action of G on M is a map

$$\begin{aligned} G \times M &\to M \\ (g, m) &\mapsto gm\,, \end{aligned}$$

which satisfies

$$\begin{aligned} em = m \quad \forall\, m \in M \qquad &\text{and} \\ g(hm) = (gh)m \quad \forall\, g, h \in G &\text{ and } \forall\, m \in M\,. \end{aligned}$$

Remark: Thus, $M \to M$, $m \mapsto gm$, is bijective for each (fixed) $g \in G$.
Can you show this?

Definition: (orbit)
The orbit of the point $m \in M$ under an action of a group G on M is defined as

$$Gm = \{gm \;:\; g \in G\}\,.$$

Remarks:

1. The orbit of a "typical" point contains $n = |G|$ elements.
2. The orbit of a "special" point contains less than $n = |G|$ elements.

Example: equilateral triangle https://youtu.be/1rUaIp5sJr8 (4 min) (6)

Definition: (stabiliser)
Let $G \times M \to M$, $(g, m) \mapsto gm$, be an action of G on M. The set of group elements that map a given $m \in M$ to itself, i.e.

$$G_m = \{g \in G \;:\; gm = m\}\,,$$

is called stabiliser (or isotropy group or little group) of m.

Remark: G_m is a group. (see exercises)

Example: equilateral triangle https://youtu.be/gPot13SMf0o (1 min) (7)

Notice that in all three cases $|Gm| \cdot |G_m| = |G|$. This is true in general for finite groups (*orbit-stabiliser theorem*, see exercises).

1.6 Conjugacy classes and normal subgroups

Definition: (conjugation)
Let G be a group. We say $x \in G$ is conjugate to $y \in G \underset{\text{Def.}}{\Leftrightarrow} \exists\, g \in G \,:\, y = gxg^{-1}$.
We then write $x \sim y$.

Show that $\sim$ is an equivalence relation, i.e. show reflexivity, symmetry and transitivity.

Examples: S_3, $\mathrm{SO}(3)$ https://youtu.be/LpBfagD302Q (6 min) (8)

Definition: (conjugacy class)
For a group G and $x \in G$ we call $\{gxg^{-1} \,:\, g \in G\}$ the conjugacy class of x.

Remarks:

1. The class of e contains only e, since $geg^{-1} = e\ \forall\, g$.
2. For abelian groups each element forms a class of its own, since $gxg^{-1} = x\ \forall\, g$.
3. In general a class is not a subgroup (cf. below).
4. Each element of G is contained in exactly one class. **Why?**
5. $|G|$ is divisible by the number of elements of each conjugacy class.
(orbit-stabiliser theorem, see exercises)
6. Later: The number of conjugacy classes is equal to the number of non-equivalent irreducible representations of a finite group.

Example: conjugacy classes of S_3 https://youtu.be/FOr3dReVKCk (3 min) (9)

Definition: (conjugate subgroups, normal subgroup)

(i) We call a subgroup $K \subseteq G$ conjugate to a subgroup $H \subseteq G$ if $\exists\, g \in G$ such that
$$K = gHg^{-1} = \{ghg^{-1} \,:\, h \in H\}\,.$$

(ii) If $ghg^{-1} \in H\ \forall\, h \in H$ und $\forall\, g \in G$ then we call H a normal subgroup (or invariant subgroup) of G.

Study the behaviour of the subgroups of S_3 under conjugation!

Remark: A finite group is called *simple* if it has no non-trivial normal subgroup.

Universität Tübingen, Fachbereich Mathematik
Dr. Stefan Keppeler Sommersemester 2021

Groups and Representations

Instruction 3 for the preparation of the lecture on 28 April 2021

1.7 Cosets and quotient groups

Definition: (coset)
Let G be a group and $H \subseteq G$ a subgroup. For $g \in G$ the set

$$gH = \{gh : h \in H\}$$

is called a left coset of H (in G). Similarly, we call $Hg = \{hg : h \in H\}$ a right coset of H.

Remarks:

1. $gH, Hg \subseteq G$.
2. If $g \in H \Rightarrow gH = Hg = H$. **Why?**
3. $|gH| = |H|$. **Why?**
4. In the following we consider mostly left cosets.
5. Two cosets g_1H and g_2H are either identical ($\Leftrightarrow g_1^{-1}g_2 \in H$) or disjoint.

   Proof: https://youtu.be/yJI8R1ju87g (2 min) (1)

6. **Can you see** that this implies that $|H|$ divides $|G|$?

Example:

S_3 and subgroups
$H_1 = \{e, (12)\}$ (not normal) and https://youtu.be/SIA9F0k1JKQ (10 min) (2)
$H_2 = \{e, (123), (132)\}$ (normal)

Definition: (quotient group)
Let H be a normal subgroup of G. We define the quotient group $(G/H, \cdot)$ as the set of cosets,

$$G/H = \{gH \ : \ g \in G\}\,, \qquad \text{with group law} \qquad (g_1H)\cdot(g_2H) = (g_1g_2)H\,.$$

Remarks:

1. $|G/H| = \frac{|G|}{|H|}$
2. $(G/H, \cdot)$ is actually a group:

   https://youtu.be/N75Wz4j_Aa8 (3 min) (3)

   Where did we need that H is normal?

Example:

https://youtu.be/GodmqXXT-pM (1 min) (4)

1.8 Direct product

Definition: (direct product)
For two groups $(A, \circ)$ and $(B, \bullet)$ the direct product is the Cartesian product $A \times B$ with group law

$$(a_1, b_1) \cdot (a_2, b_2) = (a_1 \circ a_2, b_1 \bullet b_2)\,.$$

Remarks:

1. $e_{A\times B} = (e_A, e_B)$ and $(a, b)^{-1} = (a^{-1}, b^{-1})$
2. for finite groups: $|A \times B| = |A||B|$
3. $G = A \times B$ has a normal subgroup isomorphic to A, and $G/A \cong B$ (and vice versa):

   https://youtu.be/0ppFs0QuI9w (5 min) (5)

   Caveat: In general, for a normal subgroup H of G, $G \ncong H \times (G/H)$. **Why not?**

1.9 Example: The homomorphism from SL(2, $\mathbb{C}$) to the Lorentz group

Let M be the Minkowski space, i.e. $M = \mathbb{R}^4$ with the Lorentz metric[1]

$$\|x\|^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2\,.$$

We call $x = (x_0, x_1, x_2, x_3)$ a four-vector. A (homogeneous) Lorentz transformation Λ is a linear map $M \to M$, which preserves the Lorentz metric, i.e.

$$\|\Lambda x\|^2 = \|x\|^2 \qquad \forall\, x \in M\,.$$

The Lorentz group $L = \mathrm{O}(3, 1)$ is the group of all (homogeneous) Lorentz transformations. We identify each $x \in M$ with a Hermitian 2×2 matrix:

$$X = x_0\mathbb{1} + x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3\,, \quad \text{where}$$

$$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\text{i.e.} \quad X = \begin{pmatrix} x_0 + x_3 & x_1 - \mathrm{i}x_2 \\ x_1 + \mathrm{i}x_2 & x_0 - x_3 \end{pmatrix}.$$

The σ_j are called Pauli matrices. **Convince yourself** that $\det X = \|x\|^2$.

Let's define a homomorphism from SL(2, $\mathbb{C}$) to $L = O(3, 1)$:

https://youtu.be/GRIdoIQWCVg (5 min) (6)

The homomorphism $\phi : \mathrm{SL}(2, \mathbb{C}) \to \mathrm{O}(3, 1)$ is not an ismomorphism.

- **Show** that ϕ is not injective.
- ϕ is not surjective either:

https://youtu.be/YGpT0Z7cwXU (2 min) (7)

[1] More precisely, $\|x\|^2 = d(x, x)$ with the pseudo-Riemannian metric $d(x, y) = x_0y_0 - x_1y_1 - x_2y_2 - x_3y_3$.

Universität Tübingen, Fachbereich Mathematik Sommersemester 2021
Dr. Stefan Keppeler

Groups and Representations

Instruction 4 for the preparation of the lecture on 3 May 2021

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2 Representations

2.1 Definitions

Definition: (representation)
Let G be a group and V a vector space. A representation (rep) Γ of G is a homomorphism $G \to \mathrm{GL}(V)$, i.e. into the bijective linear maps $V \to V$, i.e. in particular

$$\Gamma(g)\Gamma(h) = \Gamma(gh) \quad \forall\, g, h \in G$$

and $\Gamma(e) = \mathbb{1}$ (identity matrix/operator). We call $\dim V$ the dimension of the representation, and we will require $\dim V > 0$.

Remarks:

1. A representation is an action of G on V (in addition: linear).
2. We say that V *carries* the representation Γ, and we call V the *carrier space* (of Γ).
3. Unless otherwise stated we consider vector spaces over $\mathbb{C}$ (maybe sometimes over $\mathbb{R}$, probably never over other fields), e.g. $\mathbb{C}^n$ or $L^2(\mathbb{R}^d)$,[1] equipped with a scalar product $\langle \cdot | \cdot \rangle : V \times V \to \mathbb{C}$.
4. Choosing an orthonormal basis of V (if finite-dimensional), $\{v_j \,:\, j = 1, \ldots, d = \dim V\}$, each $\Gamma(g)$ corresponds to a $d \times d$ matrix with elements

$$\Gamma(g)_{jk} = \langle v_j | \Gamma(g) v_k \rangle\,,$$

   and we call Γ a *matrix representation.*

   We say: The v_i transform under G in the representation Γ.
5. $\dim V = \operatorname{tr} \Gamma(e)$ (if V is finite-dimensional)

Example:

$$\text{a 3-dimensional rep of } S_3 \qquad \texttt{https://youtu.be/K2Dt1BGL1Vk}\ (2\,\text{min}) \tag{1}$$

Determine $\Gamma(\underline{\times})$ and $\Gamma(\divideontimes)$.

Definition: (faithful representation)
We call a representation faithful if the homomorphism $\Gamma : G \to \mathrm{GL}(V)$ is injective, i.e. if different group elements are represented by different matrices.

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[1] It's best to think of the finite-dimensional case for the moment. In the infinite-dimensional case we'd really want separable Hilbert spaces and bounded linear operators $\Gamma(g)$.

Remarks:

1. Every group has the trivial representation, with $\Gamma(g) = \mathbb{1}\ \forall\, g \in G$; in general not faithful.
2. If G has a non-trivial normal subgroup H, then a representation of the quotient group G/H induces a representation of G. This representation is not faithful.

   https://youtu.be/PS2YTzl4a2Y (3 min) (2)

   Show: If a non-trivial rep Γ is not faithful, then G has a non-trivial normal subgroup H, and Γ induces a faithful representation of the quotient group G/H.

Definition: (unitary representation)
A representation $\Gamma : G \to \mathrm{GL}(V)$ is called unitary, if $\Gamma(g)$ is unitary $\forall\, g \in G$, i.e. $\langle \Gamma(g)v|\Gamma(g)w\rangle = \langle v|w\rangle\ \forall\, v, w \in V$.

Remarks:

1. If V is finite-dimensional and if we choose an orthonormal basis, then such a representation is in terms of unitary matrices.
2. Unitary representations are important for applications in physics, since it is in terms of them that symmetries are implemented in quantum mechanics (or quantum field theory).

2.2 Equivalent Representations

Definition: (equivalent representations)
We say that two representations $\Gamma : G \to \mathrm{GL}(V)$ and $\tilde{\Gamma} : G \to \mathrm{GL}(W)$ are equivalent, if there exists an invertible linear map $S : V \to W$ such that

$$\Gamma(g) = S^{-1}\,\tilde{\Gamma}(g)\,S \quad \forall\, g \in G\,.$$

Remark: If the linear map is unitary, i.e. (writing U instead of S) $U : V \to W$ with $\langle U\phi|U\psi\rangle_W = \langle\phi|\psi\rangle_V$ then we say that the representations are *unitarily equivalent*. For finite-dimensional representations we have $V \cong W \cong \mathbb{C}^{\dim V}$, and by choosing orthonormal bases U becomes a unitary matrix.

Theorem 2. *Let G be a finite group, $\Gamma : G \to \mathrm{GL}(V)$ a (finite-dimensional) representation and $\langle\cdot|\cdot\rangle$ a scalar product on V. Then Γ is equivalent to a unitary representation.*

Proof: $\quad (v, w) = \sum_{g\in G}\langle\Gamma(g)v|\Gamma(g)w\rangle$ is also a scalar product:

https://youtu.be/-HWa-iaBZVk (4 min) (3)

Let $\{v_j\}$ be an orthonormal basis (ONB) with respect to $\langle\cdot|\cdot\rangle$ and $\{w_j\}$ an ONB with respect to $(\cdot,\cdot)$. Then there exists an invertible map $S : V \to V$ with $Sw_j = v_j$ (change of basis). Hence

$$(v, w) = \langle Sv|Sw\rangle\,. \qquad \text{https://youtu.be/L\_HIR-Ug7nc (4 min)} \tag{4}$$

Finally, $\tilde{\Gamma}$ with $\tilde{\Gamma}(g) = S\Gamma(g)S^{-1}$ is equivalent to Γ and unitary:

https://youtu.be/1iXbQXyYvjY (6 min) (5)

2.4 Irreducible Representations

Definition: (invariant subspace)
Let $\Gamma : G \to \mathrm{GL}(V)$ be a representation and $U \subseteq V$ a subspace of V. U is called invariant subspace (with respect to Γ), if $\Gamma(g)v \in U \;\forall\, v \in U$ and $\forall\, g \in G$.

Remark: Every carrier space has two trivial invariant subspaces, namely V and $\{0\}$. All other invariant subspace (if there are any) are called non-trivial.

Definition: (irreducible representation & complete reducibility)
We call a representation $\Gamma : G \to \mathrm{GL}(V)$

(i) irreducible, if V possesses no non-trivial invariant subspace. Then we also call V irreducible with respect to Γ.

(ii) reducible, if V possesses a non-trivial invariant subspace U.

(iii) completely reducible, if V can be written as a direct sum of irreducible invariant subspaces.

Abbreviation for "irreducible representation": *irrep*

Example:

https://youtu.be/kfpZhLGZ9IA (5 min) (6)

Write $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ as linear combination of vectors from U_1 and U_2. **Construct** an ONB (with respect to the canonical scalar product) s.t. the first basis vector spans U_1 and the other two span U_2.

Groups and Representations

Instruction 5 for the preparation of the lecture on 5 May 2021

2.4 Irreducible Representations (cont.)

Theorem 3. *Let $\Gamma : G \to \mathrm{GL}(V)$ be a unitary representation and $U \subseteq V$ an invariant subspace. Then:*

(i) $U^\perp = \{v \in V \,:\, \langle u|v\rangle = 0 \quad \forall\, u \in U\}$ *is also invariant,*

(ii) *the restrictions $\Gamma|_U$ and $\Gamma|_{U^\perp}$ define representations Γ^1 and Γ^2, and*

(iii) *Γ ist equivalent to $\Gamma^1 \oplus \Gamma^2$; we simply write $\Gamma = \Gamma^1 \oplus \Gamma^2$.*

Corollary: (Maschke's Theorem)
We can write every (finite-dimensional) unitary representation as a direct sum of irreducible representations.

Explain why this implies that for finite groups every (finite-dimensional) representation is completely reducible.

Proof (of Theorem 3 & Corollary):

https://youtu.be/FJjdh6WNVF8 (4 min) (1)

Remark: Given a completely reducible representation $\Gamma : G \to \mathrm{GL}(V)$, we can find a basis of V such that in matrix notation

$$\Gamma(g) = \begin{pmatrix} \Gamma^1(g) & & & \mathbf{0} \\ & \Gamma^2(g) & & \\ & & \Gamma^3(g) & \\ \mathbf{0} & & & \ddots \end{pmatrix},$$

where the Γ^j are irreducible ($d_j \times d_j$ blocks with $d_j = \dim \Gamma^j$).

Here an irreducible representation can appear more than once, (relabel)

$$\Gamma = \underbrace{\Gamma^1 \oplus \cdots \oplus \Gamma^1}_{a_1 \text{ times}} \oplus \underbrace{\Gamma^2 \oplus \cdots \oplus \Gamma^2}_{a_2 \text{ times}} \oplus \cdots = \bigoplus_j a_j \Gamma^j \,,$$

https://youtu.be/f-9KAeE3oPc (2 min)

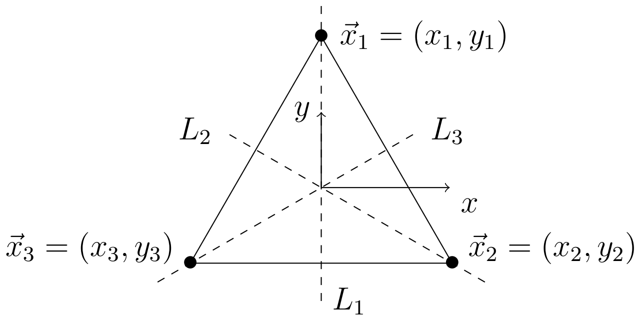
i.e. in Γ the irreducible representation Γ^j is contained a_j times.

2.4.1 Example: O_A operators for the group D_3

Consider a group G of orthogonal matrices $A : \mathbb{R}^n \to \mathbb{R}^n$ and some functions $\varphi : \mathbb{R}^n \to \mathbb{C}$. Then $(O_A\varphi)(\vec{x}) = \varphi(A^{-1}\vec{x})$ defines a representation of G on some function space.

https://youtu.be/9qYex-CZTf0 (2 min) (2)

Choose $G = D_3$, the symmetry group of an equilateral triangle ($\cong S_3$),



$n = 2$, and $\varphi_j(\vec{x}) = \mathrm{e}^{-|\vec{x}-\vec{x}_j|^2}$. Then $\mathrm{span}(\varphi_1, \varphi_2, \varphi_3)$ is invariant and carries a three-dimensional rep of S_3:

https://youtu.be/v3hSAc-h7mg (5 min) (3)

We find two invariant subspaces, one carries the trivial rep, and the other carries a two-dimensional rep:

https://youtu.be/FdpzE7YqR_k (4 min) (4)

2.5 Schur's Lemmas and orthogonality of irreps

Theorem 4. (Schur's Lemma 1)
Let G be a group, $\Gamma : G \to \mathrm{GL}(V)$ a finite-dimensional, irreducible representation and $A : V \to V$ a linear map. If A commutes with Γ, i.e. $A\Gamma(g) = \Gamma(g)A\ \forall\, g \in G$, then $A = c\mathbb{1}$ for some $c \in \mathbb{C}$.

Proof: https://youtu.be/JStDaicTKaU (3 min) (5)

Corollary: *For an abelian group G, every irreducible representation has dimension* 1. **Explain!**

Theorem 5. (Schur's Lemma 2)
Let G be a group, $\Gamma : G \to \mathrm{GL}(V)$ and $\tilde{\Gamma} : G \to \mathrm{GL}(W)$ two finite-dimensional, unitary irreducible representations and $A : V \to W$ a linear map. If

$$A\Gamma(g) = \tilde{\Gamma}(g)A \quad \forall\, g \in G\,,$$

then $A = 0$ or Γ and $\tilde{\Gamma}$ are unitarily equivalent.

Proof: https://youtu.be/Or9gnVwmuDo (5 min) (6)

Universität Tübingen, Fachbereich Mathematik
Dr. Stefan Keppeler Sommersemester 2021

Groups and Representations

Instruction 6 for the preparation of the lecture on 10 May 2021

2.5 Schur's Lemmas and orthogonality of irreps (cont.)

Theorem 6. *Let G be a finite group and Γ^j, $j = 1, 2, \ldots$, non-equivalent unitary irreducible representations with $\dim \Gamma^j = d_j$. Then the matrix elements obey the orthogonality relation*

$$\frac{1}{|G|} \sum_{g \in G} \overline{(\Gamma^j(g)_{\mu\nu})}\, \Gamma^k(g)_{\mu'\nu'} = \frac{1}{d_j}\, \delta_{jk}\, \delta_{\mu\mu'}\, \delta_{\nu\nu'}$$

$\forall\ \mu, \nu = 1, \ldots, d_j$ and $\forall\ \mu', \nu' = 1, \ldots, d_k$.

Proof: https://youtu.be/vWhHL-2cCTw (13 min) (1)

Corollary 1 to Theorem 6:

$$\sum_j d_j^2 \leq |G| \qquad \text{https://youtu.be/bHY8dAFQA-c (4 min)} \tag{2}$$

Remark: Later we will see that we actually have equality.

2.6 Characters

Definition: (character)
For a finite-dimensional representation $\Gamma : G \to \mathrm{GL}(V)$ we call $\chi : G \to \mathbb{C}$ with

$$\chi(g) = \operatorname{tr} \Gamma(g)$$

the character of the representation.

Remarks:

1. In terms of matrix elements we have $\chi(g) = \sum_{\mu=1}^{\dim V} \Gamma(g)_{\mu\mu}$.
2. Equivalent reps have the same characters.
3. Characters are constant on conjugacy classes.

Show remarks 2 and 3.

Corollary 2 to Theorem 6. *Let G be a finite group and Γ^j, $j = 1, 2, \ldots$, non-equivalent, irreducible representations with $\dim \Gamma^j = d_j$. Then the characters $\chi^j = \operatorname{tr} \Gamma^j$ obey the orthogonality relation*

$$\frac{1}{|G|} \sum_{g \in G} \overline{\chi^j(g)}\, \chi^k(g) = \delta_{jk}\,.$$

Proof: https://youtu.be/q1P7YKGsFWg (2 min) (3)

Remarks:

1. Since the characters depend only on the conjugacy class, we can rewrite the orthogonality relation as

$$\frac{1}{|G|}\sum_c n_c\,\overline{\chi_c^j}\,\chi_c^k = \delta_{jk}\,,$$

where c labels classes and n_c is the number of elements in class c.

2. Let m be the number of different conjugacy classes of G, and let p the number of non-equivalent irreducible representations. Then

$$p \leq m\,. \qquad \text{https://youtu.be/6OMuqu1iMNk (3 min)}$$

In the exercises you will show that we actually have $p = m$.
The $m \times m$ matrix with entries χ_c^j is called *character table* of the group.

3. If Γ is irreducible then

$$\frac{1}{|G|}\sum_{g\in G}|\chi(g)|^2 = 1\,.$$

If Γ is reducible then

$$\frac{1}{|G|}\sum_{g\in G}|\chi(g)|^2 > 1\,.$$

https://youtu.be/aewkyIA009c (4 min) (4)

In https://youtu.be/FdpzE7YqR_k we encountered three reps of S_3.
Check for irreducibility!

Example: Here's another irrep of $D_3 \cong S_3$:

$$\Gamma(e) = \Gamma(C) = \Gamma(\bar{C}) = 1\,, \qquad \Gamma(\sigma_1) = \Gamma(\sigma_2) = \Gamma(\sigma_3) = -1$$

Hence, the character table of $D_3 \cong S_3$ reads

		$\{e\}$	$\{C,\bar{C}\}$	$\{\sigma_1,\sigma_2,\sigma_3\}$
trivial rep	χ^1	1	1	1
other 1D rep	χ^2	1	1	-1
2D irrep	χ^3	2	-1	0

Remarks:

4. If $\Gamma = \bigoplus_j a_j \Gamma^j$ with irreps Γ^j then

$$a_j = \frac{1}{|G|}\sum_c n_c\,\overline{\chi_c^j}\,\chi_c\,. \qquad \text{https://youtu.be/yW5um6C10k4 (2 min)} \qquad (5)$$

Use this in order to verify that the 3D rep of $D_3 \cong S_3$ from https://youtu.be/FdpzE7YqR_k is a direct sum of the trivial rep of the 2D irrep.

Supplement: D_3-reps from https://youtu.be/FdpzE7YqR_k.
3D rep:

$$\Gamma(e) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \Gamma(C) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \Gamma(\bar{C}) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\Gamma(\sigma_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \Gamma(\sigma_2) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \Gamma(\sigma_3) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

2D irrep:

$$\Gamma^3(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \Gamma^3(C) = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \qquad \Gamma^3(\bar{C}) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

$$\Gamma^3(\sigma_1) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \Gamma^3(\sigma_2) = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \qquad \Gamma^3(\sigma_3) = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

Groups and Representations

Instruction 7 for the preparation of the lecture on 12 May 2021

2.7 The regular representation

Definition: (group algebra)
For a finite group G we define its group algebra $\mathcal{A}(G)$ as the vector space spanned by the group elements, i.e. we form linear combinations

$$\mathcal{A}(G) \ \ni \ r = \sum_{j=1}^{|G|} r_j g_j \,, \quad r_j \in \mathbb{C} \,,$$

with multiplication rule

$$\left(\sum_{j=1}^{|G|} q_j g_j \right) \left(\sum_{k=1}^{|G|} r_k g_k \right) = \sum_{j=1}^{|G|} \sum_{k=1}^{|G|} q_j r_k \, g_j g_k \,.$$

induced by group multiplication.

Remarks:

https://youtu.be/3QjW40hhVag (3 min) (1)

Now we can write group multiplication as

$$g g_j = \sum_{k=1}^{|G|} g_k \, R(g)_{kj} \,,$$

where $R(g)_{kj}$ encodes the group table: For g and j fixed, $R(g)_{kj} = 1$ for exactly one value of k and it vanishes for all others. R defines a representation of G on $\mathcal{A}(G)$, the so-called *regular representation*:

https://youtu.be/xgF-ifx0sgg (4 min) (2)

Example:

regular rep R of S_3 https://youtu.be/Hbd2obSIQ1g (5 min) (3)

Determine $R(\overline{\times})$ with the same choice of basis as in the video.

Theorem 7. *The regular representation R contains all irreps of G, and the multiplicity of irrep Γ^j is given by its dimension d_j,*

$$R = \bigoplus_{j=1}^{p} d_j \, \Gamma^j \qquad \begin{pmatrix} p & = & \textit{number of non-equivalent} \\ & & \textit{irreducible representations} \end{pmatrix} .$$

Remark: Hence, there exists a regular matrix S such that

$$S^{-1}\,R(g)\,S = \begin{pmatrix} 1 & & & & & & \\ & \Gamma^2(g) & & & & & \\ & & \ddots & & & & \\ & & & \Gamma^2(g) & & & \\ & & & & \ddots & & \\ & & & & & \Gamma^p(g) & \\ & & & & & & \ddots & \\ & & & & & & & \Gamma^p(g) \end{pmatrix}.$$

$$\underbrace{\qquad\qquad}_{d_2 \text{ blocks}} \cdots \underbrace{\qquad\qquad}_{d_p \text{ blocks}}$$

Proof:

$$\text{https://youtu.be/-QPH0cyBLpc (4 min)} \tag{4}$$

Corollary. *We have* $\sum_{j=1}^{p} d_j^2 = |G|$.

Show this!

2.8 Product representations and Clebsch-Gordan coefficients

Definition: (product representation)
For representations $\Gamma^\mu : G \to \mathrm{GL}(U)$ and $\Gamma^\nu : G \to \mathrm{GL}(V)$ we define the product representation $\Gamma^{\mu\otimes\nu} : G \to \mathrm{GL}(U \otimes V)$ by

$$\Gamma^{\mu\otimes\nu}(g) = \Gamma^\mu(g) \otimes \Gamma^\nu(g) \quad \forall\, g \in G\,.$$

Remarks:

1. $\Gamma^{\mu\otimes\nu}$ is a representation:

$$\text{https://youtu.be/s8aENniin5Y (3 min)} \tag{5}$$

2. For the characters we have

$$\chi^{\mu\otimes\nu}(g) = \operatorname{tr}\Gamma^{\mu\otimes\nu}(g) = \operatorname{tr}\left(\Gamma^\mu(g)\otimes\Gamma^\nu(g)\right) = \operatorname{tr}\Gamma^\mu(g)\,\operatorname{tr}\Gamma^\nu(g) = \chi^\mu(g)\chi^\nu(g)\,.$$

3. Even for irreducible Γ^μ and Γ^ν the product representation is in general reducible,

$$\Gamma^\mu \otimes \Gamma^\nu = \bigoplus_\lambda a_\lambda \Gamma^\lambda \quad \text{with} \quad d_\mu d_\nu = \sum_\lambda a_\lambda d_\lambda\,.$$

   According to character orthogonality the multiplicities are

$$a_\lambda = \frac{1}{|G|}\sum_c n_c\,\overline{\chi_c^\lambda}\,\chi_c^\mu \chi_c^\nu\,.$$

Example: $\mathbb{Z}_2 \cong \{e, P\}$ has two one-dimensional irreps. Character table:

	e	P
$\chi^1 = \Gamma^1$	1	1
$\chi^2 = \Gamma^2$	1	-1

Construct the regular rep R.
Reduce $\Gamma = R \otimes R$ to a direct sum of irreps.

Clebsch-Gordan coefficients. In our live session we will go through some awkward looking but frequently used notation in the context of the basis change from a product basis to basis in which subsets of the basis vectors span irreducible subspaces.

Recap: Tensor products

Let U and V be vector spaces with bases $\{u_i\}$ and $\{v_j\}$, respectively, and let $W = U \otimes V$ with basis $\{w_k\}$, where $w_k = u_i \otimes v_j$. Further let $A : U \to U$ and $B : V \to V$ be linear maps. Then $D := A \otimes B$ is the linear map $W \to W$ with

$$Dw_k = Au_i \otimes Bv_j\,, \quad \text{where } k = (i, j)\,,$$

by linearity extended to arbitrary $w \in W$, i.e. for $w = \sum_k \alpha_k w_k = \sum_{ij} \alpha_{ij} u_i \otimes v_j$ we have

$$Dw = \sum_{i,j} \alpha_{ij}\, Au_i \otimes Bv_j\,.$$

In matrix components:

$$\begin{aligned} Au_i &= \sum_{i'} u_{i'} A_{i'i}\,, \qquad Bv_j = \sum_{j'} v_{j'} B_{j'j} \quad \text{and} \\ Dw_k &= \sum_{k'} w_{k'} D_{k'k} = \sum_{i'j'} (u_{i'} \otimes v_j) A_{i'i}\, B_{j'j}\,, \end{aligned}$$

i.e. $D_{k'k} \equiv D_{i'j'ij} = A_{i'i}\, B_{j'j}$. If everything is finite-dimensional then

$$\operatorname{tr} D = \sum_k D_{kk} = \sum_{i,j} A_{ii} B_{jj} = \operatorname{tr} A \cdot \operatorname{tr} B = \operatorname{tr}(A \otimes B)\,.$$

Scalar products on U and V induce a scalar product on W by

$$\langle w_k | w_{k'} \rangle = \langle u_i | u_{i'} \rangle_U\, \langle v_j | v_{j'} \rangle_V\,,$$

again extended by (sesqui-)linearity.
If $\{u_i\}$ and $\{v_j\}$ are ONB with respect to $\langle\,|\,\rangle_U$ and $\langle\,|\,\rangle_V$, then $\{w_k\}$ is also orthonormal,

$$\langle w_k | w_{k'} \rangle = \delta_{ii'} \delta_{jj'} = \delta_{kk'}\,.$$

Groups and Representations

2.8 Clebsch-Gordan coefficients

In general we can decompose $W = U \otimes V$ into a direct sum of (under G) invariant irreducible subspaces W^λ_α, with $\dim(W^\lambda_\alpha) = d_\lambda$. The index $\alpha = 1, \ldots, a_\lambda$ distinguishes different subspaces carrying the same irreducible representation, i.e. $\exists\, U$, such that

$$U^{-1}\,\Gamma^{\mu\otimes\nu}\,U = \begin{pmatrix} \boxed{\Gamma^1} & & & & & & & \\ & \ddots & & & & & & \\ & & \boxed{\Gamma^1} & & & & & \\ & & & \ddots & & & & \\ & & & & \boxed{\Gamma^\lambda} & & & \\ & & & & & \ddots & & \\ & & & & & & \boxed{\Gamma^\lambda} & \\ & & & & & & & \ddots \end{pmatrix} .$$

$$\underbrace{\qquad\qquad\qquad}_{a_1 \text{ blocks}} \quad \cdots \quad \underbrace{\qquad\qquad\qquad\qquad}_{a_\lambda \text{ blocks}}$$

Thus U provides the change of basis from the $\{w_k\}$ to some new basis $\{w^\lambda_{\alpha\ell}\}$ in which the representation matrices are block-diagonal. Here $\ell = 1, \ldots, d_\lambda$ numbers the absis vectors of W^λ_α.

By choosing ONBs on both sides U becomes unitary.

Remark: In general U is highly non-unique.

The rest is essentially **notation** – somewhat nasty, but widely used, and sometimes even useful.

With $k = (i, j)$ and in so-called Dirac notation, one writes

$$|w^\lambda_{\alpha\ell}\rangle = \sum_{ij} |w_{ij}\rangle \underbrace{\langle i, j(\mu,\nu)\alpha, \lambda, \ell\rangle}_{\text{Clebsch-Gordan coefficients}} . \qquad (*)$$

The *Clebsch-Gordan coefficients* are matrix elements of U, with

(i, j): row index (old basis),

(α, λ, ℓ): column index (new basis),

(μ, ν): fix. (Tells us which product is decomposed.)

The inverse of $(*)$ is

$$|w_{ij}\rangle = \sum_{\alpha\lambda\ell} |w^\lambda_{\alpha\ell}\rangle\langle\alpha,\lambda,\ell(\mu,\nu)i,j\rangle\,,$$

(this defines $\langle\alpha,\lambda,\ell(\mu,\nu)i,j\rangle$)
and with U unitary we have $\langle\alpha,\lambda,\ell(\mu,\nu)i,j\rangle = \overline{\langle i,j(\mu,\nu)\alpha,\lambda,\ell\rangle}$

- The CG coefficients satisfy "orthonormality and completeness relations"

$$\begin{aligned}\sum_{\alpha\lambda\ell}\langle i',j'(\mu,\nu)\alpha,\lambda,\ell\rangle\langle\alpha,\lambda,\ell(\mu,\nu)i,j\rangle &= \delta_{i'i}\delta_{j'j} \qquad \text{and}\\ \sum_{ij}\langle\alpha',\lambda',\ell'(\mu,\nu)i,j\rangle\langle i,j(\mu,\nu)\alpha,\lambda,\ell\rangle &= \delta_{\alpha'\alpha}\delta_{\lambda'\lambda}\delta_{\ell'\ell}\,,\end{aligned}$$

  in matrix notation $U^\dagger U = \mathbb{1} = UU^\dagger$.

- **simplified notation**
  - $|i,j\rangle := |w_{ij}\rangle$ and $|\alpha,\lambda,\ell\rangle := |w^\lambda_{\alpha\ell}\rangle$
  - Einstein summation convention (sum over repeated indices)
  - $\langle i,j(\mu,\nu)\alpha,\lambda,\ell\rangle = \langle i,j|\alpha,\lambda,\ell\rangle$

  Then we can write

$$\begin{aligned}\Gamma^{\mu\otimes\nu}(g)|i,j\rangle &= |i',j'\rangle\Gamma^\mu(g)_{i'i}\Gamma^\nu(g)_{j'j} \qquad \text{and}\\ \Gamma^{\mu\otimes\nu}(g)|\alpha,\lambda,\ell\rangle &= |\alpha,\lambda,\ell'\rangle\Gamma^\lambda(g)_{\ell'\ell}\,,\end{aligned}$$

  and conclude

$$\begin{aligned}\langle\alpha',\lambda',\ell'|\Gamma^{\mu\otimes\nu}(g)|\alpha,\lambda,\ell\rangle &= \langle\alpha',\lambda',\ell'|\alpha,\lambda,\ell''\rangle\Gamma^\lambda(g)_{\ell''\ell} = \delta_{\alpha'\alpha}\delta_{\lambda'\lambda}\delta_{\ell'\ell''}\Gamma^\lambda(g)_{\ell''\ell}\\ &= \delta_{\alpha'\alpha}\delta_{\lambda'\lambda}\Gamma^\lambda(g)_{\ell'\ell}\\ &\underset{(*)}{=} \langle\alpha',\lambda',\ell'|\Gamma^{\mu\otimes\nu}(g)|i,j\rangle\langle i,j|\alpha,\lambda,\ell\rangle\\ &= \langle\alpha',\lambda',\ell'|i',j'\rangle\Gamma^\mu(g)_{i'i}\Gamma^\nu(g)_{j'j}\langle i,j|\alpha,\lambda,\ell\rangle\,.\end{aligned}$$

  (relation between elements of the representation matrices in the old and the new basis)

Example:
In quantum mechanics (the spin degree of freedom of) a spin-$\frac{1}{2}$ particle is described by a vector in $\mathbb{C}^2$. The basis vectors

$$|\uparrow\rangle := \begin{pmatrix}1\\0\end{pmatrix} \qquad \text{and} \qquad |\downarrow\rangle := \begin{pmatrix}0\\1\end{pmatrix}$$

transform in a two-dimensional representation of SU(2), namely $\Gamma^2(g) = g \;\forall\; g \in \mathrm{SU}(2)$.
Consider two spin-$\frac{1}{2}$ particles: $\mathbb{C}^2 \otimes \mathbb{C}^2 \cong \mathbb{C}^4$, spanned by the product basis

$$|\uparrow\uparrow\rangle := |\uparrow\rangle \otimes |\uparrow\rangle\,, \qquad |\uparrow\downarrow\rangle := |\uparrow\rangle \otimes |\downarrow\rangle\,, \qquad |\downarrow\uparrow\rangle := |\downarrow\rangle \otimes |\uparrow\rangle\,, \qquad |\downarrow\downarrow\rangle := |\downarrow\rangle \otimes |\downarrow\rangle\,,$$

transforms in $\Gamma^{2\otimes 2}$. Define a new basis,

$$|0,0\rangle := \frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}}\,, \qquad |1,1\rangle := |\uparrow\uparrow\rangle\,, \qquad |1,0\rangle := \frac{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}{\sqrt{2}}\,, \qquad |1,-1\rangle := |\downarrow\downarrow\rangle\,.$$

In the exercises we show:

- $|0,0\rangle$ transforms in the spin-0 representation of SU(2) (one-dimensional – trivial representation), and
- $|1,m\rangle$, $m = -1,0,1$, transform in the spin-1 representation (three-dimensional) of SU(2).

Clebsch-Gordan coefficients:

	$\vert\uparrow\uparrow\rangle$	$\vert\uparrow\downarrow\rangle$	$\vert\downarrow\uparrow\rangle$	$\vert\downarrow\downarrow\rangle$
$\langle 0,0\vert$	0	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	0
$\langle 1,1\vert$	1	0	0	0
$\langle 1,0\vert$	0	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	0
$\langle 1,-1\vert$	0	0	0	1

i.e. e.g. $\langle 1,0|\uparrow\downarrow\rangle = \frac{1}{\sqrt{2}}$.

In general one labels the unitary irreducible representations of SU(2) by their so-called spin quantum number $s \in \frac{1}{2}\mathbb{N}_0$; the correspong representation has dimension $2s+1$.

Universität Tübingen, Fachbereich Mathematik Sommersemester 2021
Dr. Stefan Keppeler

Groups and Representations

Instruction 8 for the preparation of the lecture on 17 May 2021

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3 Applications in quantum mechanics

3.1 Expansion in irreducible basis functions and selections rules

Setting:

L^2-spaces & unitary operators https://youtu.be/5yJbnMbWZK4 (2 min) (1)

Lemma 8. *Let G be a (finite) group of unitary linear operators $V \to V$, $A \in G$,*[1] *and let $\psi_1^\nu, \dots, \psi_{d_\nu}^\nu$ be functions that transform in the unitary irreducible representation Γ^ν (with $\dim(\Gamma^\nu) = d_\nu$), i.e.*

$$A\psi_\alpha^\nu = \sum_{\beta=1}^{d_\nu} \psi_\beta^\nu \, \Gamma^\nu(A)_{\beta\alpha} \, .$$

Then $\exists\, C_\nu \in \mathbb{C}$ such that $\langle \psi_\alpha^\nu | \psi_\beta^\mu \rangle = C_\nu \, \delta_{\nu\mu} \, \delta_{\alpha\beta}$.

Proof:

https://youtu.be/Ru30m0wiOTM (7 min) (2)

Remarks:

https://youtu.be/i-KGixakDDM (3 min) (3)

Have you heard the term *selection rule* before? If not, never mind. If yes, in which context? Let's speak about it in the live session.

3.2 Invariance of the Hamiltonian and degeneracies

A special role is played by the *Hamiltonian* $H : V \to V$ (a linear self-adjoint operator) of a quantum mechanical system. In particular, its eigenvalues are the possible *energy levels* in which we can find the system.

Let H be the Hamiltonian of a quantum mechanical system and let A be a unitary operator. If

$$AH = HA \, ,$$

we say that A commutes with the Hamiltonian or that A *leaves H invariant.*

The set of all symmetry operations (realised by unitary operators) that leave H invariant forms a group G, the *symmetry group* of H. **Why is this a group?**

Every eigenspace of H, say $\{\psi \in V : H\psi = E\psi\}$, carries a representation of the symmetry group G:

https://youtu.be/t0_HAXTVgmg (2 min) (4)

[1] Alternatively, view the operators A as unitary representation of a group G on V.

This representation can, in principle, be reducible or irreducible; typically it is irreducible: Consider now an invariant subspace with $H\psi \in U \; \forall \, \psi \in U$. Then:

- If U is irreducible then all $\psi \in U$ have the same energy:

$$\texttt{https://youtu.be/hdODKxaR4Ec} \; (3\,\text{min}) \tag{5}$$

- States transforming in different irreps can have different energies – at least, symmetry does not force them to have the same energy:

$$\texttt{https://youtu.be/er1ZQ3a8\_38} \; (2\,\text{min}) \tag{6}$$

If states transforming in different irreps still have the same energy, we speak about "accidental degeneracy". Possible reasons:

- "Fine-tuning" of one or several parameters in H (unlikely).
- We haven't correctly identified the full symmetry group, i.e. we have overlooked some symmetry.

Conclusions:

- Degenerate states to a given energy typically transform in an irrep of the symmetry group of H, i.e. they can be classified by irreps.
- number of degenerate states = dimension of the irrep

Example (& outlook): Hydrogen atom

$$\texttt{https://youtu.be/fYfsXfVIh3E} \; (8\,\text{min}) \tag{7}$$

Groups and Representations

Instruction 9 for the preparation of the lecture on 19 May 2021

3.3 Perturbation theory and lifting of degeneracies

Setting: Hamiltonian is a sum of a (known) term H_0 and a (small) perturbation H',

$$H = H_0 + H' \,.$$

Let G be the symmetry group of H_0. Two possibilities:

1. H' is also invariant under G.
2. H' is only invariant under a subgroup $B \subset G$.

In case 1 the spectra of H_0 and of H look similar (same multiplicities).

Case 2 (symmetry breaking) typically leads to a splitting of energy levels:

- Eigenstates of H transform in irreps of B.
- Degenerate eigenstates of H_0 transform in irreps of G.
- Eigenspaces of H_0 carry reps of B, in general reducible.
  States transforming in different irreps of B, in general, have different energies.
  States transforming in the same irrep of B, are still degenerate.

https://youtu.be/_IDScHV5Jps (3 min) (1)

Examples:

1. Hydrogen atom as in Section 3.2.
   Adding a small radially symmetric potential $V(r)$ (but not $\frac{1}{r}$) breaks the O(4)-symmetry to O(3). Each energy level splits into n levels with different ℓ. Each new level is still $(2\ell{+}1)$-fold degenerate.

   https://youtu.be/y_tIHpehjcY (2 min) (2)

2. Fine structure of hydrogen.
   - Take electron spin into account: instead of $L^2(\mathbb{R}^3)$ consider $L^2(\mathbb{R}^3) \otimes \mathbb{C}^2$.
   - Intermediate step: Consider $H \otimes \mathbb{1}_2$. States which so far transformed in irrep $\Gamma^{2\ell+1}$ of O(3), now transform in rep $\Gamma^{2\ell+1} \otimes \Gamma^2$ of SU(2), but energies are unchanged, only the degeneracy is doubled.

     Wait, why SU(2)*?* https://youtu.be/2dFq2LwrrMU (4 min) (3)

   - Now add the perturbation H', containing spin-dependent terms (spin-orbit coupling), but still invariant under SU(2). Splittings follow from

     $$\Gamma^{2\ell+1} \otimes \Gamma^2 = \Gamma^{2\ell} \oplus \Gamma^{2\ell+2} \,.$$

     https://youtu.be/p1SZsPfGjEM (7 min) (4)

---

4 Expansion into irreducible basis vectors

4.1 Projection operators onto irreducible bases

Recall Lemma 8 and the following remark about constructing irreducible invariant subspaces. Let's elaborate on this idea. Let U be a (completely reducible) representation (e.g. by unitary operators) on V and let $e_1^\nu, \ldots, e_{d_\nu}^\nu \in V$ be functions/vectors that transform in the unitary irreducible representation Γ^ν (with $\dim(\Gamma^\nu) = d_\nu$). We can expand every $\psi \in V$ into such basis vectors, i.e.

$$\psi = \sum_\mu \sum_{\beta=1}^{d_\mu} c_\beta^\mu \, e_\beta^\mu \,,$$

with expansion coefficients $c_\beta^\mu \in \mathbb{C}$. Let's apply $U(g)$:

$$\texttt{https://youtu.be/ZA1qsZNHl5M} \text{ (6 min)} \tag{5}$$

This motivates the following definition.

Definition: (generalised projection operators)
Let G be a group, U a representation, Γ^μ an irreducible representation, $\dim \Gamma^\mu = d_\mu$. We call

$$P_{jk}^\mu = \frac{d_\mu}{|G|} \sum_{g \in G} [\Gamma^\mu(g)^{-1}]_{jk} \, U(g)$$

generalised projection operator.

Remark: In the following Γ will always be unitary, i.e.

$$[\Gamma^\mu(g)^{-1}]_{jk} = [\Gamma^\mu(g)^\dagger]_{jk} = \overline{\Gamma^\mu(g)_{kj}} \qquad \text{(cf. above)}.$$

We will study the properties of these operators on the next instruction sheet.

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Dr. Stefan Keppeler

Groups and Representations

Instruction 10 for the preparation of the lecture on 31 May 2021

4.1 Projection operators onto irreducible bases (cont.)

Theorem 9. (Properties of P^μ_{jk}) *With the above definitions we have:*

(i) *For fixed $\psi \in V$ and for fixed μ and j the d_μ vectors $P^\mu_{jk}\psi$, $k = 1, \dots, d_\mu$, either all vanish or they transform in irrep Γ^μ, i.e. $U(g)P^\mu_{jk} = \sum_\ell P^\mu_{j\ell}\Gamma^\nu(g)_{\ell k}$.*

(ii) $P^\mu_{ji}P^\nu_{\ell k} = \delta_{\mu\nu}\delta_{jk}P^\mu_{\ell i}$.

(iii) $P^\mu_j = P^\mu_{jj}$ *is a projection operator.*

(iv) $P^\mu = \sum_j P^\mu_j$ *is a projection operator onto the invariant subspace U^μ containing all vectors transforming in the irreducible representation Γ^μ.*

(v) $\sum_\mu P^\mu = \mathbb{1}$ *if V completely reducible (here always assumed).*

(vi) $U(g) = \sum_\mu \sum_{j,k} \Gamma^\mu(g)_{kj}P^\mu_{jk}$ *(inversion of definition).*

Proof:

(i) & (ii)	`https://youtu.be/Xenr0VXpvcM`	(4 min)	(1)
(iii)–(v)	`https://youtu.be/05OMW7Cao8w`	(2 min)	(2)
(vi)	`https://youtu.be/M-4KmZHsM0w`	(2 min)	(3)

Examples:

1. Reduction of span(ϕ_1, ϕ_2, ϕ_3) from Section 2.4.1 (invariant under $D_3 \cong S_3$). S_3 has two 1-dimensional and one 2-dimensional irrep $(\Gamma^1, \Gamma^2, \Gamma^3)$.

   generalised projection operators `https://youtu.be/laouieOnL4A` (6 min) (4)

   Apply to some vector, say ϕ_1:

$\mu = 1, 2$	`https://youtu.be/nMMHx7_zs_w`	(3 min)	(5)
$\mu = 3$	`https://youtu.be/8sDomkziGvA`	(5 min)	(6)

2. Reducing a product representation:

   `https://youtu.be/79QuhXEDkGY` (3 min) (7)

4.2 Irreducible operators and the Wigner-Eckart Theorem

Definition: (irreducible operators)
Let G be a group, $U : G \to \mathrm{GL}(V)$ a representation, and Γ^μ a unitary irreducible representation with $\dim \Gamma^\mu = d_\mu$. A set of linear operators $O_i^\mu : V \to V$, $i = 1, \ldots, d_\mu$, which transform under G as follows,

$$U(g)\, O_i^\mu\, U(g)^{-1} = \sum_{j=1}^{d_\mu} O_j^\mu\, \Gamma^\mu(g)_{ji}\,,$$

is called a set of irreducible operators corresponding to irrep Γ^μ.
(The O_i^μ are also called irreducible tensors or irreducible tensor operators).

Remarks:

1. The definition makes sense:

$$\texttt{https://youtu.be/KEr1n5iC394}\ (4\,\mathrm{min}) \tag{8}$$

2. Special case: If Γ^μ is the trivial representation then the operator O^μ (no index i, since $d_\mu = 1$) commutes with $U(g)\ \forall\, g \in G$, cf. Section 3.2.
3. If O_i^μ, $i = 1, \ldots, d_\mu$, are irreducible operators and if $|e_j^\nu\rangle$, $j = 1, \ldots, d_\nu$, are irreducible basis vectors, then the vectors $O_i^\mu |e_j^\nu\rangle$ transform in the product rep $\Gamma^{\mu\otimes\nu}$.
   Show this!
   We can reduce this product representation (cf. Section 2.8) and expand the vectors $O_i^\mu |e_j^\nu\rangle$ in the irreducible basis $\{|w_{\alpha\ell}^\lambda\rangle\}$,

$$O_i^\mu |e_j^\nu\rangle = \sum_{\alpha\lambda\ell} |w_{\alpha\ell}^\lambda\rangle\langle \alpha, \lambda, \ell(\mu,\nu) i, j\rangle\,. \tag{$*$}$$

   This leads to…

Theorem 10. (Wigner-Eckart)
Let O_i^μ be irreducible operators and let $|e_j^\nu\rangle$ be irreducible vectors. Then

$$\langle e_\ell^\lambda | O_i^\mu | e_j^\nu\rangle = \sum_\alpha \langle \alpha, \lambda, \ell(\mu,\nu) i, j\rangle\, \langle \lambda \| O^\mu \| \nu\rangle_\alpha$$

with the so-called reduced matrix element (which isn't a matrix element…)

$$\langle \lambda \| O^\mu \| \nu\rangle_\alpha = \frac{1}{d_\lambda} \sum_k \langle e_k^\lambda | w_{\alpha k}^\lambda\rangle\,.$$

Can you prove this, using $(*)$ and the proof of Lemma 8?

Universität Tübingen, Fachbereich Mathematik
Dr. Stefan Keppeler Sommersemester 2021

Groups and Representations

Instruction 11 for the preparation of the lecture on 2 June 2021

4.2 Irreducible operators and the Wigner-Eckart Theorem (cont.)

Remarks (on Wigner-Eckart):

https://youtu.be/DnBKfkmI4R8 (4 min) (1)

Example: time-dependent perturbation theory for dipole radiation

https://youtu.be/iTP7E2z54F8 (8 min) (2)

4.3 Left ideals and idempotents

The generalised projection operators allow us to decompose reducible reps into sums of irreps. To this end we already have to know the irreps.

Question: How to construct the irreps?

Idea: Reduce the regular rep (see Instruction 7), as it contains all irreps Γ^μ

Recall: Carrier space is the group algebra $\mathcal{A}(G) = \mathrm{span}(g_1, \dots, g_{|G|})$.

Definition: (left ideal, minimal left ideal)
A subspace $L \subseteq \mathcal{A}(G)$ that is invariant under left multiplication is called *left ideal*, i.e.

$$r \in L \text{ and } q \in \mathcal{A}(G) \quad \Rightarrow \quad qr \in L\,.$$

A left ideal L is called *minimal* if it does not contain any non-trivial left ideal $K \subset L$.

Remarks:

1. One similarly defines right ideals and two-sided ideals. (We use only left ideals.)
2. L is a left ideal $\Leftrightarrow$ L is an invariant subspace.
3. L is a minimal left ideal $\Leftrightarrow$ L is an irreducible invariant subspace.

Show remarks 2 and 3.

Denote by P^μ_α the projection operator onto the minimal left ideal L^μ_α, i.e. $P^\mu_\alpha \mathcal{A}(G) = L^\mu_\alpha$.
As before μ labels the non-equivalent irreps, and $\alpha = 1, \dots, d_\mu$. Demand the following. . .

. . . properties of P^μ_α:

(i) $P^\mu_\alpha r \in L^\mu_\alpha \quad \forall\, r \in \mathcal{A}(G)$.

(ii) If $q \in L^\mu_\alpha$ then $P^\mu_\alpha q = q$.

(iii) $P^\mu_\alpha P^\nu_\beta = \delta_{\mu\nu}\delta_{\alpha\beta} P^\mu_\alpha$.

It then follows that

(iv) $P^\mu_\alpha q = q P^\mu_\alpha \quad \forall\, q \in \mathcal{A}(G)$.

Proof: https://youtu.be/ebET50quvPk (3 min) (3)

We define $L^\mu = \bigoplus_\alpha L^\mu_\alpha$ and first construct the projection operator P^μ onto L^μ:

$$\texttt{https://youtu.be/YLY-j-LNkH8}\ (4\,\text{min}) \tag{4}$$

Lemma 11. *P^μ is given by right multiplication with e_μ, i.e. $P^\mu q = q e_\mu\ \forall\ q \in \mathcal{A}(G)$.*

Remarks:

1. P^μ is linear.
2. $e_\mu e_\nu = \delta_{\mu\nu} e_\mu$ – cf. property (iii). **Show this.**
3. With $e = \sum_{\mu,\alpha} e^\mu_\alpha$ this also works for projections to minimal left ideals, $P^\mu_\alpha q = q e^\mu_\alpha$.

Definition: (idempotents)
An element $e_\mu \in \mathcal{A}(G)$ that satisfies $e_\mu^2 = e_\mu$ is called (an) idempotent. If $e_\mu^2 = \xi_\mu e_\mu$ for some non-zero $\xi_\mu \in \mathbb{C}$ then we call e_μ essentially idempotent.

Remarks:

1. We say the idempotent e_μ *generates* the left ideal L^μ, i.e. $L^\mu = \{q e_\mu\ :\ q \in \mathcal{A}(G)\}$.
2. An idempotent is called *primitive*, if it generates a minimal left ideal. Otherwise it can be written as a sum $e_1 + e_2$ of two non-zero idempotents with $e_1 e_2 = 0 = e_2 e_1$.

Theorem 12.
The idempotent e_μ is primitive. $\Leftrightarrow$ For every $q \in \mathcal{A}(G)\ \exists\, \lambda_q \in \mathbb{C}$ s.t. $e_\mu q e_\mu = \lambda_q e_\mu$.

Proof:

$$\texttt{https://youtu.be/jrqF23SpENg}\ (10\,\text{min}) \tag{5}$$

Theorem 13. *The left ideals generated by two primitive idempotents, e_1 and e_2, carry equivalent irreps Γ^1 and Γ^2 iff $e_1 q e_2 \neq 0$ for at least one $q \in \mathcal{A}(G)$.*

Proof:

$$\texttt{https://youtu.be/Wy3NS9IE\_oY}\ (9\,\text{min}) \tag{6}$$

Example: The primitive idempotent

$$e_1 = \frac{1}{|G|} \sum_{j=1}^{|G|} g_j$$

generates the one-dimensional left ideal L^1, which carries the trivial representation.
Show this!

Universität Tübingen, Fachbereich Mathematik
Dr. Stefan Keppeler Sommersemester 2021

Groups and Representations

Instruction 12 for the preparation of the lecture on 7 June 2021

4.3.1 Dimensions and characters of the irreducible representations

Theorem 14. *Let G be a group with group algebra $\mathcal{A}(G)$, and let*

$$e_\mu = \sum_{g \in G} a_g \, g$$

be a primitive idempotent with corresponding left ideal $L^\mu = \mathcal{A}(G)e_\mu$, carrying irrep Γ^μ with $\dim \Gamma^\mu = d_\mu$. Then $\forall \, h \in G$

$$\chi^\mu(h) = \operatorname{tr} \Gamma^\mu(h) = \frac{|G|}{n_c} \sum_{g \in c} \overline{a_g}$$

where c is the conjugacy class of h with n_c elements.

Proof:

https://youtu.be/iSiwLsOS8w8 (12 min) (1)

---

5 Representations of the symmetric group and Young diagrams

5.1 One-dimensional irreps and associate representations of S_n

The *alternating group* A_n is the group of even permutations of $\{1, 2, \ldots, n\}$ (i.e. each element is the product of an even number of transpositions). A_n is a normal subgroup of S_n, with quotient group $S_n/A_n \cong \mathbb{Z}_2$.

$\Rightarrow$ S_n has two one-dimensional representations, induced by the by the representations of $\mathbb{Z}_2$ (cf. Problems 8, 14 & 15):

$$\Gamma^{\mathrm{s}}(p) = 1 \quad \forall \, p \in S_n \text{ (trivial representation) and}$$
$$\Gamma^{\mathrm{a}}(p) = \operatorname{sgn}(p) = \begin{cases} 1 & \text{for } p \text{ even} \\ -1 & \text{for } p \text{ odd} \end{cases} .$$

$\operatorname{sgn}(p)$ is called sign or parity of the permutation p.

Later we will see: There are no other one-dimensional representations of S_n.

Lemma 15. *The symmetriser $s = \sum_{p \in S_n} p$ and the anti-symmetriser $a = \sum_{p \in S_n} \operatorname{sgn}(p) p$ are essentially idempotent and primitive.*

Prove this!

Corresponding irreps: https://youtu.be/syAbXy1vExo (4 min) (2)

Show the non-equivalence of these two irreps using Theorem 13.

Definition: (associate representations)
For a representation Γ^λ of S_n with dimension d_λ, we call Γ^λ and $\widetilde{\Gamma^\lambda} = \Gamma^\lambda \otimes \Gamma^{\mathrm{a}}$ associate representations.

Remarks:

1. Γ^{s} and Γ^{a} are associate to each other.
2. $\dim(\widetilde{\Gamma^\lambda}) = d_\lambda$
3. $\widetilde{\Gamma^\lambda}$ is irreducible $\Leftrightarrow$ Γ^λ is irreducible.
   Why? Recall the irreducibility criterion from Instruction 6.
4. If $\chi^\lambda(p) = 0$ for all odd p, then $\widetilde{\Gamma^\lambda}$ is equivalent to Γ^λ (**why?**), and Γ^λ is called *self-associate*. Otherwise they are non-equivalent.

Theorem 16. *Let Γ^λ and Γ^μ be irreps of S_n. Then*

(i) $\Gamma^\lambda \otimes \Gamma^\mu$ *contains* Γ^{s} *exactly once (not at all), if* Γ^λ *and* Γ^μ *are equivalent (non-equivalent).*

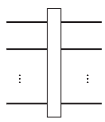
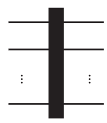
(ii) $\Gamma^\lambda \otimes \Gamma^\mu$ *contains* Γ^{a} *exactly once (not at all), if* Γ^λ *and* Γ^μ *are associate (not associate).*

Proof:

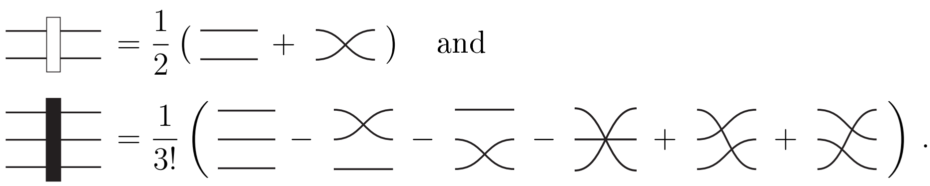
https://youtu.be/0qKaJx422ng (7 min) (3)

5.1.1 Some more birdtracks

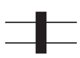
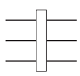
In birdtrack notation we denote symmetrisers and anti-symmetrisers by open and solid bars, respectively, i.e.

$\frac{1}{n!}s = \frac{1}{n!}\sum_{p\in S_n} p =$  and $\frac{1}{n!}a = \frac{1}{n!}\sum_{p\in S_n} \mathrm{sgn}(p)\, p =$  .

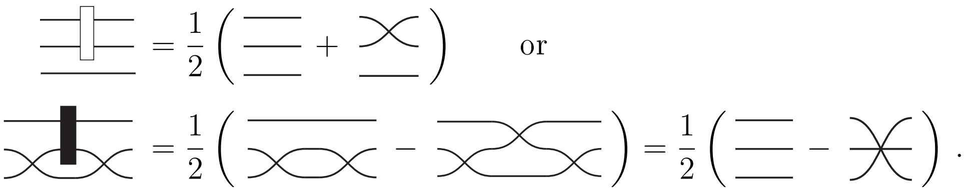
Note that we include a factor of $\frac{1}{n!}$ in the definition of bars over n lines. For instance,

 (4)

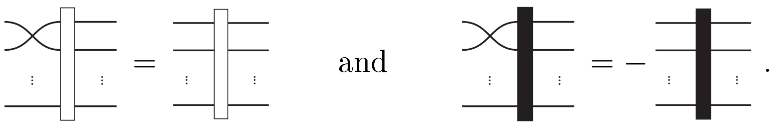
Notice that in birdtrack notation the sign of a permutation, $(-1)^K$, is determined by the number K of line crossings; if more than two lines cross in a point, one should slightly perturb the diagram before counting, e.g.  $\rightsquigarrow$  (K=3).

Expand  and  as in (4).

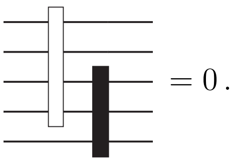
We also use the corresponding notation for partial (anti-)symmetrisation over a subset of lines, e.g.



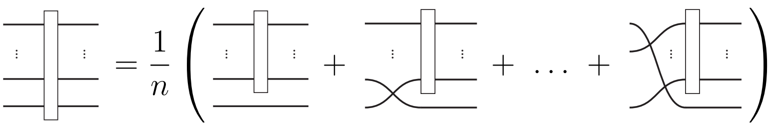
It follows directly from the definition of S and A that when intertwining any two lines S remains invariant and A changes by a factor of (-1), i.e.

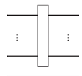


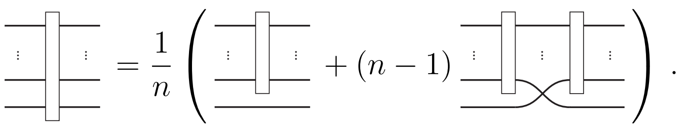
Explain why this implies that whenever two (or more) lines connect a symmetriser to an anti-symmetrizer the whole expression vanishes, e.g.


(5)

Symmetrisers and anti-symmetrisers can be built recursively. To this end notice that on the r.h.s. of



we have sorted the terms according to where the last line is mapped – to the nth, to the $(n-1)$th, ..., to the first line line. Multiplying with  from the left and disentangling lines we obtain the compact relation


(6)

Derive the corresponding recursion relation for anti-symmetrisers.

Universität Tübingen, Fachbereich Mathematik
Dr. Stefan Keppeler Sommersemester 2021

Groups and Representations

Instruction 13 for the preparation of the lecture on 9 June 2021

5.2 Young diagrams and Young tableaux

Definition: (partition, Young diagram)
A partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ of a natural number n is a (finite) sequence of positive integers with

$$\sum_{i=1}^{r} \lambda_i = n \quad \text{and} \quad \lambda_i \geq \lambda_{i+1} \,.$$

Let λ and μ be two partitions for the same n.

(i) We say that λ and μ are equal, if $\lambda_i = \mu_i \;\forall\, i$.

(ii) We say $\lambda > \mu$ if the first non-vanishing term of the sequence $\lambda_i - \mu_i$ is positive.

Graphically, a partition can be represented by a *Young diagram*:

https://youtu.be/zSsEYZqiCcM (4 min) (1)

Each partition corresponds to a conjugacy class of S_n and vice versa:

https://youtu.be/JMiaRRXWxaU (2 min) (2)

Since the number of Young diagrams with n boxes is equal to the number of conjugacy classes of S_n, it is also equal to the number of non-equivalent irreps of S_n.

Further definitions: *Young tableau*, *normal* Young tableau, and *standard* Young tableau:

https://youtu.be/3oOSaYheyRg (3 min) (3)

The normal Young tableau corresponding to partition λ we denote by Θ_λ. We obtain an arbitrary tableau from Θ_λ by a permutation p of the numbers in the boxes:

$$\Theta_\lambda^p = p\Theta_\lambda \,.$$

This implies $q\Theta_\lambda^p = \Theta_\lambda^{qp}$. **Example:**

$$\Theta_{\boxplus}^{(23)} = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} \qquad \text{since} \qquad \Theta_{(2,2)} = \Theta_{\boxplus} = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}$$

Write down all standard tableaux for S_4.

5.3 Young operators

Definitions: Let Θ^p_λ be a Young tableau.
A *horizontal* permutation h^p_λ permutes only numbers within rows of Θ^p_λ.
A *vertical* permutation v^p_λ permutes only numbers within columns of Θ^p_λ.
Furthermore, we define

the *(row-)symmetriser* $$s^p_\lambda = \sum_{\{h^p_\lambda\}} h^p_\lambda \,,$$

the *(column-)anti-symmetriser* $$a^p_\lambda = \sum_{\{v^p_\lambda\}} \mathrm{sgn}(v^p_\lambda)\, v^p_\lambda \quad \text{and}$$

the *Young operator* (or irreducible symmetriser) $$e^p_\lambda = s^p_\lambda\, a^p_\lambda = \sum_{\{h^p_\lambda\}} \sum_{\{v^p_\lambda\}} \mathrm{sgn}(v^p_\lambda)\, h^p_\lambda\, v^p_\lambda \,.$$

Example: standard tableaux for S_3

https://youtu.be/pqOOq2mWiLc (6 min) (4)

Expressed in birdtracks:

https://youtu.be/F19O19xUrdE (3 min) (5)

Verify that $e_{\yng(2,1)}$ is essentially idempotent. Try both, birdtracks and cycle notation.

Observations:

1. For each tableau Θ^p_λ the horizontal and the vertical permutations, $\{h^p_\lambda\}$ and $\{v^p_\lambda\}$, form subgroups of S_n, with $\{h^p_\lambda\} \cap \{v^p_\lambda\} = \{e\}$.
   We obtain the subgroups for Θ^p_λ from those for Θ_λ by conjugation with p, hence $e^p_\lambda = p\, e_\lambda\, p^{-1}$.

2. s^p_λ and a^p_λ are (total) symmetriser and anti-symmetriser of the corresponding subgroup, in the sense that
   $$s^p_\lambda h^p_\lambda = h^p_\lambda s^p_\lambda = s^p_\lambda \qquad \text{and} \qquad a^p_\lambda v^p_\lambda = v^p_\lambda a^p_\lambda = \mathrm{sgn}(v^p_\lambda) a^p_\lambda \,.$$

3. s^p_λ and a^p_λ are essentially idempotent, but in general not primitive.
   The e^p_λ are essentially idempotent and primitive (here for S_3, later for S_n).
   Can you show primitivity for $e_{\yng(2,1)}$?

4. $e_{\yng(3)} = s$ and $e_{\yng(1,1,1)} = a$ generate the two one-dimensional irreps of S_3 (cf. Section 5.1).
   $e_{\yng(2,1)}$ generates a two-dimensional (minimal) left ideal of $\mathcal{A}(S_3)$:

   https://youtu.be/gX6Q7HzJzSE (6 min) (6)

   $\Rightarrow$ The Young operators of the normal Young tableaux generate all irreps of S_3.

5. **Determine** the (minimal) left ideal generated by $e^{(23)}_{\yng(2,1)}$.

6. **Verify** that $e = \frac{1}{6} e_{\yng(3)} + \frac{1}{3} e_{\yng(2,1)} + \frac{1}{3} e^{(23)}_{\yng(2,1)} + \frac{1}{6} e_{\yng(1,1,1)}$ and conclude that the regular rep of S_3 is completely reduced by the Young operators of the standard Young tableaux.

Groups and Representations

Instruction 14 for the preparation of the lecture on 14 June 2021

5.4 Irreducible representations of S_n

Theorem 17. *Let $\lambda \neq \mu$ be partitions of $n \in \mathbb{N}$.*

(i) *The Young operators e_λ^p are essentially idempotent, i.e. $(e_\lambda^p)^2 = \eta_\lambda e_\lambda^p$ with $\eta_\lambda \neq 0$,*

(ii) *the $\frac{1}{\eta_\lambda} e_\lambda^p$ are primitive idempotents.*

(iii) *The irreducible representations generated by e_λ and e_μ are not equivalent.*

(iv) *The irreducible representations generated by e_λ and e_λ^p are equivalent.*

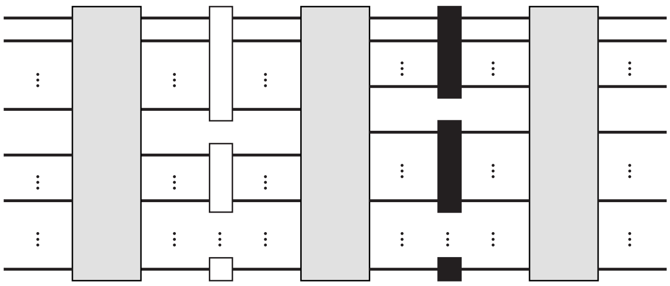
Proof: First notice that no two terms in

$$e_\lambda = \sum_{\{h_\lambda\}} \sum_{\{v_\lambda\}} \mathrm{sgn}(v_\lambda) h_\lambda v_\lambda$$

are proportional to the same permutation. **Why?** In particular, $e_\lambda \neq 0$ and

$$e_\lambda = e + \text{terms proportional to } p \in S_n \backslash \{e\} \,.$$

In birdtracks we have:

$e_\lambda^p =$  (1)

https://youtu.be/0e-rjjzijJw (2 min)

With this we can prove all four statements:

(iii)	https://youtu.be/Kq_Z6mnpbXE (7 min)
(i)	https://youtu.be/wEVE7g9w74Y (8 min)
(ii) & (iv)	https://youtu.be/wrh1ILmhthE (4 min)

Remark: Unfortunately, for $n \geq 5$ the Young operators for the standard tableaux no longer satisfy $e_\lambda^p e_\lambda^q = 0 \; \forall \, p \neq q$ (they still satisfy $e_\lambda^p e_\mu^q = 0 \; \forall \, \lambda \neq \mu$, see (iii) above). However, the ideals generated by the Young operators of the standard tableaux are still linearly independent (see exercises) and

$$\mathcal{A}(S_n) = \bigoplus_{\left\{ \substack{\text{standard} \\ \text{tableaux}} \, \Theta_\lambda^p \right\}} \mathcal{A}(S_n) e_\lambda^p \, .$$

(without proof). In particular, this implies that $\dim \left(\mathcal{A}(S_n) e_\lambda^p \right)$ is given by the number of standard tableaux for the partition λ.

5.5 Calculating characters using Young diagrams

The dimension d_λ of irrep Γ^λ is given by the number of standard tableaux for the partition λ. The *hook length formula* (which we won't prove) is very convenient:

$$d_\lambda = \frac{n!}{\prod_{i,j} h_{ij}} \, . \qquad \texttt{https://youtu.be/DxPI8QOlh\_Q} \; (3\,\text{min}) \tag{2}$$

Determine the dimensions of all irreps of S_4.

Before calculating characters we introduce the notion of a *skew hook:*

https://youtu.be/E_ahyAWIhp0 (2 min) (3)

Here's a recipe (without proof) for calculating characters. Let c be a conjugacy class of S_n with disjoint cycles of lengths $a_1, a_2, \ldots, a_q$. Recursively determine the character χ_c^λ as follows:

- Choose any cycle of c, say with length a_i.
- Denote by $\bar{c}$ the class of S_{n-a_i}, obtained by removing the cycle a_i from c.
- For the Young diagram Θ_λ determine all skew hooks of length a_i and denote the Young diagram(s) of S_{n-a_i}, obtained by removing such a skew hook by $\Theta_{\bar{\lambda}}$. Then

  $$\chi_c^\lambda = \sum_{\bar{\lambda}} \pm \chi_{\bar{c}}^{\bar{\lambda}}$$

  with "+" for positive skew hooks and "−" for negative skew hooks.
- Iterate this procedure.
- If no box of the Young diagram remains then $\chi_{()}^{\bar{\lambda}=0} = 1$.
  (Don't forget the sign of the last skew hook removed!)
- If there is no skew hook of length a_i then $\chi_c^\lambda = 0$.

Example: https://youtu.be/XnSE5E6m6fg (7 min) (4)

Determine the characters of the irrep of S_3 corresponding to .

Explain how we recover the number of standard tableaux when recursively determining the character of the identity.

Universität Tübingen, Fachbereich Mathematik Sommersemester 2021
Dr. Stefan Keppeler

Groups and Representations

Instruction 15 for the preparation of the lecture on 16 June 2021

6 Lie groups

When speaking about infinite groups we will combine the notion of a group with notions from others areas of mathematics. There will be precise definitions using notions like "topological space", "connectedness" or "differentiable manifold". However, we will not introduce all these notions and concepts in detail. If you are familiar with these notions – fine. If not, don't panic! Some of the subtelties will not be relevant for the cases we are interested in, so we will gloss over them. Aspects which are important in our context will be introduced and discussed carefully, such that no prior knowledge beyond, say, multivariable calculus/analysis in $\mathbb{R}^n$ will be required.

6.1 Topological groups

Definition: (topological group)
A group $(G, \circ)$ is called topological group if

(i) G is a topological space,

(ii) the map $G \ni g \mapsto g^{-1} \in G$ is continuous, and

(iii) the map $G \times G \ni (g, h) \mapsto g \circ h \in G$ is continuous.

Remark: Unless otherwise stated, our topology will be the standard topology on $\mathbb{R}^n$ or $\mathbb{C}^n$, and the induced topology on subspaces.

Examples:

$\mathrm{GL}(n)$, $\mathrm{O}(n)$, $\mathrm{U}(n)$ etc. https://youtu.be/0b_GhknIuNY (3 min) (1)

Definition: (isomomorphism)
Two topological groups G and H are called isomorphic, if there exists a bijective map $f : G \to H$, which is both, an isomorphism of groups, and a homeomorphism of topological spaces (i.e. f is continuous and f^{-1} is continuous).

(Non-)Example:

https://youtu.be/tDCZNBbqYRk (7 min) (2)

Definition: (homogeneous space)
A topological space X is called homogeneous, if for every pair $x, y \in X$ there exists a homeomorphism $f : X \to X$ s.t. $f(x) = y$.

Remark: Every topological group G is homogeneous. This is nice when studying *local* properties.

https://youtu.be/CYu-XpNKu3Q (2 min) (3)

Interesting *global* properties are *compactness* and *connectedness*.

Examples (compactness):

1. $\mathrm{O}(n)$ is compact:

$$\texttt{https://youtu.be/8pFew9Cp-10}\ (3\,\text{min}) \tag{4}$$

2. $\mathrm{O}(1,1)$ is not compact:

$$\texttt{https://youtu.be/zvWRGgcJVpY}\ (2\,\text{min}) \tag{5}$$

3. $\mathrm{GL}(n,\mathbb{R})$ is not compact. **Why?**

Definition: (connected component)
The connected component of $g \in G$ is the union of all connected sets that contain g.

Remarks:

1. A connected component is actually connected.
2. Let $G_0 \subseteq G$ be the connected component of the identity e.
   Show that G_0 is a subgroup of G.
   G_0 is a normal subgroup, and the quotient group G/G_0 is totally disconnected:

$$\texttt{https://youtu.be/-\_mTHYztXUw}\ (4\,\text{min}) \tag{6}$$

Examples:

1. $\mathrm{SU}(2)$ is (simply) connected, since with the parametrisation of Problem 19,

$$\begin{aligned} &\mathrm{SU}(2) \ni \ g = \begin{pmatrix} u & -\overline{v} \\ v & \overline{u} \end{pmatrix} , \\ &|u|^2 + |v|^2 = 1 \quad \Leftrightarrow \quad (\mathrm{Re}\,u)^2 + (\mathrm{Im}\,u)^2 + (\mathrm{Re}\,v)^2 + (\mathrm{Im}\,v)^2 = 1 , \end{aligned}$$

   $\mathrm{SU}(2)$ is homeomorphic to S^3, and spheres S^n with $n \geq 2$ are (simply) connected.
2. $\mathrm{O}(n)$ is not connected. **Why?**

6.2 Example: SO(2)

Before discussing Lie groups in general, let's look at an example, which illustrates some of the basic ideas. We'll do this in our live session.

Groups and Representations

Instruction 16 for the preparation of the lecture on 21 June 2021

6.3 Lie groups

Definition: (Lie group)
A group $(G, \circ)$ is called Lie group if

(i) G is an analytic manifold,
(ii) the map $G \ni g \mapsto g^{-1} \in G$ is analytic, and
(iii) the map $G \times G \ni (g, h) \mapsto g \circ h \in G$ is analytic.

Remarks:

1. An n-dimensional manifold M is something that locally looks like a piece of $\mathbb{R}^n$:

   https://youtu.be/HQMI050AEjw (3 min) (1)

2. Locally, group elements are analytic functions of n parameters:

   https://youtu.be/_VqV9uK76oo (3 min) (2)

3. The so-called *structure constants* $c^j_{k\ell}$ *of the Lie group* are determined by the group law:

   https://youtu.be/4mzmPRyOjgE (6 min) (3)

   Properties of the structure constants:

   (i) For abelian groups $c^j_{k\ell} = 0$, since then $f(x, y) = f(y, x)$.
   (ii) $c^j_{k\ell} = -c^j_{\ell k}$
   (iii) $\sum_\ell (c^j_{k\ell} c^\ell_{nm} + c^j_{n\ell} c^\ell_{mk} + c^j_{m\ell} c^\ell_{kn}) = 0$

   The last identity follows from associativity of group multiplication by comparing the third order terms in the coordinate expansions of $g(h\tilde{g})$ and $(gh)\tilde{g}$.

Examples: matrix Lie groups

1. $\mathrm{GL}(n, \mathbb{R})$ is a Lie group:

   https://youtu.be/so7fTTzjsLo (4 min) (4)

2. For $\mathrm{GL}(n, \mathbb{C})$ consider real and imaginary part of the matrix elements as coordinates and argue as before (in terms of submanifolds of $\mathbb{R}^{2n^2}$).
3. For groups like $\mathrm{O}(n)$, $\mathrm{U}(n)$, $\mathrm{SO}(n)$ or $\mathrm{SU}(n)$ one first observes that they are closed subgroups of $\mathrm{GL}(n, \mathbb{R})$ or $\mathrm{GL}(n, \mathbb{C})$, respectively. One can show that closed subgroups of Lie groups are Lie (sub-)groups. (Later we will study some of these more explicitly.)

6.4 Lie algebras

Definition: A Lie algebra $\mathfrak{g}$ is a vector space over a field K (here mostly $\mathbb{R}$, sometimes $\mathbb{C}$), with an operation

$$\begin{aligned}[\cdot,\cdot] : \mathfrak{g}\times\mathfrak{g} &\to \mathfrak{g}\\ (X,Y) &\mapsto [X,Y]\end{aligned}$$

called *Lie bracket*, which satisfies ($\forall\ X,Y,Z\in\mathfrak{g}$):

(i) $[\lambda X+\mu Y, Z] = \lambda[X,Z]+\mu[Y,Z] \qquad \forall\ \lambda,\mu\in K$ (linearity)

(ii) $[X,Y] = -[Y,X]$ (anti-symmetry)

(iii) $[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]=0$ (Jacobi identity)

Remarks:

1. A Lie algebra is called commutative if $[X,Y]=0\ \forall\ X,Y\in\mathfrak{g}$.

2. One can show that the tangent space to a Lie group G at the identity is a Lie algebra $\mathfrak{g}$.

To this end consider curves $g(t)$ in G with $g(0)=e$. Then the derivative (in a chart) at $t=0$ is a tangent vector.

For matrix Lie groups we can explicitly define the Lie algebra elements, as matrices:

$$-\mathrm{i}\dot{g}(0) := -\mathrm{i}\frac{\mathrm{d}g}{\mathrm{d}t}(0)\in\mathfrak{g}\,.$$

The Lie bracket is now the matrix commutator (rather times $(-\mathrm{i})$, see below)

$$[X,Y]=XY-YX\,.$$

The commutator is linear and anti-symmetric, the Jacobi identity can be verified by direct calculation.

It remains to show that $X,Y\in\mathfrak{g}$ implies that also $(-\mathrm{i})[X,Y]\in\mathfrak{g}$.

https://youtu.be/6VsahKnoDHY (8 min) (5)

3. Choosing a basis $\{X_j\}$ of $\mathfrak{g}$ we have

$$[X_j,X_k]=\mathrm{i}\sum_\ell c^\ell_{jk}X_\ell$$

with the *structure constants* c^ℓ_{jk} *of the Lie algebra* (basis dependent).

The structure constants of the Lie algebra are equal to the structure constants of the corresponding the Lie group (see Section 6.3) – supposing an appropriate choice of basis and coordinates: As basis $\{X_j\}$ for $\mathfrak{g}$ choose the tangent vectors to the coordinate lines in a chart $U\ni e$, i.e. for matrix Lie groups in an explicit parametrisation by taking derivatives with respect to the parameters,

$$\begin{aligned}&X_j=-\mathrm{i}\dot{g}(0)\quad\text{with}\quad g(t)=\varphi^{-1}(0,\dots,0,x_j=t,0,\dots,0)\,,\\ &\text{hence}\quad X_j=-\mathrm{i}\frac{\partial\varphi^{-1}}{\partial x_j}(0)\,.\end{aligned}$$

In Section 6.3 we compared expansions of gh and hg, here we essentially expanded $hgh^{-1}-g$. Properties (ii) & (iii) of the structure constants of Section 6.3 now follow from the Lie bracket properties (ii) & (iii) of the commutator.

Universität Tübingen, Fachbereich Mathematik
Dr. Stefan Keppeler Sommersemester 2021

Groups and Representations

Instruction 17 for the preparation of the lecture on 23 June 2021

6.4 Lie Algebras (cont.)

Let us consider special curves though $e \in G$, namely one-parameter subgroups, i.e. solutions of

$$\dot{g}(t) = \mathrm{i}X g(t)\,, \quad g(0) = e\,,$$

with $X \in \mathfrak{g}$. We write $g(t) = \exp(\mathrm{i}Xt)$. For matrix Lie groups this exponential is given by the absolutely and uniformly convergent series (cf. Problem 29)

$$\exp(\mathrm{i}tX) = \sum_{\nu=0}^{\infty} \frac{(\mathrm{i}t)^\nu}{\nu!} X^\nu$$

For the special groups with $\det g = 1$ the generators are traceless, since

$$\det g(t) = \det(\mathrm{e}^{\mathrm{i}tX}) = \mathrm{e}^{\mathrm{i}t \,\mathrm{tr}\, X} \stackrel{!}{=} 1 \qquad \Leftrightarrow \qquad \mathrm{tr}\, X = 0\,.$$

For unitary groups, i.e. $gg^\dagger = \mathbb{1}$, the generators are Hermitian, since

$$g(t)^\dagger = g(t)^{-1} \quad \Leftrightarrow \quad \mathrm{e}^{-\mathrm{i}tX^\dagger} = \mathrm{e}^{-\mathrm{i}tX} \quad \Leftrightarrow \quad X = X^\dagger\,.$$

Examples:

1. SO(3), defining rep, Lie algebra, structure constants:

   https://youtu.be/5ud3zMg-epo (5 min) (1)

2. O_A operators for SO(3):

   https://youtu.be/Ia7SX4PrmNo (7 min) (2)

6.5 More on SO(3)

Parametrise rotations as $R_{\vec{n}}(\psi)$, with rotation angle ψ and rotation axis $\vec{n}$:

https://youtu.be/VBitdDXs9XQ (4 min) (3)

Topology of SO(3):

https://youtu.be/bUu6amDkNb0 (3 min) (4)

Further observation: Rotations about a fixed axis form a (one-parameter) subgroup of SO(3). Such a subgroup is isomorphic to SO(2) (cf. Section 6.2). For arbitrary rotations $R \in \mathrm{SO}(3)$ we have (by explicit calculation using the generators from above)

$$R R_{\vec{n}}(\psi) R^{-1} = R_{\vec{n}'}(\psi) \qquad \text{with} \quad \vec{n}' = R\vec{n}\,.$$

This implies that all rotations by the same angle are in the same conjugacy class.

Universität Tübingen, Fachbereich Mathematik — Sommersemester 2021
Dr. Stefan Keppeler

Groups and Representations

Instruction 18 for the preparation of the lecture on 28 June 2021

6.6 Invariant integration: Haar measure

When representing finite groups we often used the rearrangement lemma as follows

$$\sum_{g\in G} f(g) = \sum_{g\in G} f(hg) = \sum_{g\in G} f(gh) \qquad \forall\, h \in G\,.$$

For continuous groups we would like to replace $\sum_{g\in G} f(g)$ by an integral, say, $\int_G f(g)\,\mathrm{d}\mu(g)$. To this end we need an invariant measure μ.

Theorem 18. (Haar measure)
Every compact topological group possesses a left- and right-invariant measure μ, called Haar measure; it is unique up to normalisation.

(in this generality without proof, but we will explicitly construct μ for compact Lie groups)

Remarks:

1. Invariance means $\quad \mu(gA) = \mu(Ag) = \mu(A) \quad \forall\, g \in G$ and all Borel sets $A \subset G$.
   Shorthand notation: $\mathrm{d}\mu(gh) = \mathrm{d}\mu(hg) = \mathrm{d}\mu(g) \quad \forall\, h \in G$.
   Why does this make sense?

2. For compact groups we will normalise such that $\quad \operatorname{vol} G = \int_G \mathrm{d}\mu(g) = 1\,.$

3. Hence, e.g. for continuous functions,

$$\int_G f(g)\,\mathrm{d}\mu(g) = \int_G f(hg)\,\mathrm{d}\mu(g) = \int_G f(gh)\,\mathrm{d}\mu(g) \qquad \forall\, h \in G\,. \tag{1}$$

   https://youtu.be/Nx-2sfc_2ro (2 min)

4. Moreover, $\displaystyle\int_G f(g^{-1})\,\mathrm{d}\mu(g) = \int_G f(g)\,\mathrm{d}\mu(g).$

   https://youtu.be/bRobQky1UQ4 (3 min) (2)

5. Existence implies uniqueness:

   https://youtu.be/keYUKEBcENk (2 min) (3)

6. One also finds invariant measures under weaker conditions. For instance locally compact groups (like $\mathrm{GL}(n,\mathbb{R})$ or the Lorentz group) possess left-invariant and right-invariant measures (unique up to normalisation) but in general the two measures are not identical.

6.6.1 Calculating the Haar measure for a Lie group

Parametrise the group elements using $n = \dim G$ parameters, i.e.[1] $g = g(x_1, \ldots, x_n)$. Then, locally,

$$\mathrm{d}\mu(g) = \varrho(x_1, \ldots, x_n)\, \mathrm{d}^n x$$

with a suitable density $\varrho(x)$ and Lebesgue measure $\mathrm{d}^n x = \mathrm{d}x_1 \ldots \mathrm{d}x_n$.

Hence, under reparametrisation $x = f(y)$ we have:

$$\varrho(x)\, \mathrm{d}^n x = \varrho\big(f(y)\big) \underbrace{\left|\det\left(\frac{\partial f}{\partial y}(y)\right)\right|}_{\text{Jacobian}} \mathrm{d}^n y =: \tilde{\varrho}(y)\, \mathrm{d}^n y$$

We now construct ϱ such that invariance holds.

To this end expand $(-\mathrm{i}) g(x)^{-1} \frac{\partial g}{\partial x_j}(x)$ in a basis $\{X_k\}$ of the Lie algebra $\mathfrak{g}$,

$$g(x)^{-1} \frac{\partial g}{\partial x_j}(x) = \mathrm{i} \sum_k X_k A(x)_{kj}\,. \tag{4}$$

https://youtu.be/-8CiMiXZW0A (4 min)

Claim: The density $\varrho(x) = |\det A(x)|$ defines a left-invariant measure.

Proof:

(i) We first check the behaviour under a (local) change of coordinates $x = f(y)$.

https://youtu.be/0XxGT09iTF4 (6 min) (5)

(ii) Given a parametrisation in a neighbourhood of g, near $\tilde{g} = hg$ we choose the parametrisation $\tilde{g}(x) = h\, g(x)$. Then $\tilde{\varrho} = \varrho$.

https://youtu.be/kGaGW0p4uvs (5 min) (6)

(iii) Any other parametrisation in a neighbourhood of $\tilde{g}$ can be achieved by a further change of coordinates as in (i). $\square$

What about right-invariance?

Near $\tilde{g} = gh$ choose the parametrisation $\tilde{g}(x) = g(x)h$. Then

$$\mathrm{d}\mu(gh) = |\det \varphi(h)|\, \mathrm{d}\mu(g)\,. \qquad \text{https://youtu.be/Waks9edvVJY (6 min)} \tag{7}$$

The factor $|\det \varphi(h)|$ is called *modular function* of G. If $|\det \varphi(h)| = 1\ \forall\ h \in G$, we say that G is unimodular, and the left-invariant measure is also right-invariant.

Compact Lie groups are unimodular.

Show this by studying $\displaystyle\int_G f(gh)\, \mathrm{d}\mu(g)$ for a constant function.

Example: Construct the Haar measure of SO(2) using the parametrisation of Sec. 6.2.

[1] Actually $g = \varphi^{-1}(x_1, \ldots, x_n)$ but here we prefer this shorthand notation.

Universität Tübingen, Fachbereich Mathematik
Dr. Stefan Keppeler Sommersemester 2021

Groups and Representations

Instruction 19 for the preparation of the lecture on 30 June 2021

6.7 Properties of compact Lie groups

Theorems 2 and 6 for representations of finite groups also hold for continuous representations of compact Lie groups, if in statements and proofs we replace

$$\frac{1}{|G|}\sum_{g\in G}\dots \qquad \text{by} \qquad \int_G \dots \,\mathrm{d}\mu(g)\,,$$

Hence:

(i) Every finite-dimensional representation is equivalent to a unitary representation.

(ii) Matrix elements of unitary irreps Γ^μ, Γ^ν (non-equivalent for $\mu \neq \nu$) are orthogonal,

$$\int_G \overline{\Gamma^\mu(g)_{jk}}\,\Gamma^\nu(g)_{j'k'}\,\mathrm{d}\mu(g) = \frac{1}{d_\mu}\delta_{\mu\nu}\delta_{jj'}\delta_{kk'}\,,$$

with $d_\mu = \dim\Gamma^\mu$.

(iii) Similarly for the characters $\chi^\mu(g) = \operatorname{tr}\Gamma^\mu(g) = \sum_j \Gamma^\mu(g)_{jj}$,

$$\int_G \overline{\chi^\mu(g)}\,\chi^\nu(g)\,\mathrm{d}\mu(g) = \delta_{\mu\nu}\,.$$

This implies again

$$\Gamma \text{ is irreducible} \quad \Leftrightarrow \quad \int_G |\chi(g)|^2\,\mathrm{d}\mu(g) = 1\,,$$

as well as: If Γ is a directe sum of irreps, $\Gamma = \bigoplus_\mu a_\mu\Gamma^\mu$, then

$$a_\mu = \int_G \overline{\chi^\mu(g)}\,\chi(g)\,\mathrm{d}\mu(g)\,.$$

For finite groups we also showed completeness of the representation matrices' elements (Problem 16) and complete reducibility of the regular representation, carried by the group algebra $\mathcal{A}(G)$ (Section 2.7). This implied that there were only finitely many non-equivalent irreps.

Similarly one can show that compact Lie groups have countably many non-equivalent (continuous) irreducible representations, which are all of finite dimension. Moreover, every continuous representation is a direct sum of irreducible representations. All this follows from the *Peter-Weyl theorem*.

Consider the vector $L^2(G)$ of functions $\phi : G \to \mathbb{C}$, with scalar product

$$\langle\phi|\psi\rangle = \int_G \overline{\phi(g)}\,\psi(g)\,\mathrm{d}\mu(g)\,.$$

The role of the regular representation is played by Γ defined as

$$\big(\Gamma(h)\phi\big)(g) = \phi(h^{-1}g) \qquad \forall\, h \in G\,.$$

Convince yourself that Γ is a representation.

Does it make sense that functions $\phi : G \to \mathbb{C}$ now play the role that elements of $\mathcal{A}(G)$ played for finite groups?

Theorem 19. (Peter-Weyl)
Let G be a compact Lie group with non-equivalent irreps Γ^μ, $\dim\Gamma^\mu = d_\mu$. Then the matrix elements $\sqrt{d_\mu}\,\Gamma^\mu(g)_{jk}$, $j,k = 1,\ldots,d_\mu$, form a complete set of orthonormal functions for $L^2(G)$.

(without proof)

Remarks:

1. We can thus expand every function $\phi \in L^2(G)$ as

$$\phi(g) = \sum_{\mu,j,k} c_{\mu jk}\,\Gamma^\mu(g)_{jk} \qquad \text{with} \qquad c_{\mu jk} = d_\mu \int_G \overline{\Gamma^\mu(g)_{jk}}\,\phi(g)\,\mathrm{d}\mu(g)$$

   (convergence in L^2-sense).

   What does this reduce to for $G = \mathrm{SO}(2) \cong \mathrm{U}(1)$? (cf. Section 6.2)

2. In physics notation we write completeness as

$$\sum_{\mu,j,k} d_\mu\,\Gamma^\mu(g)_{jk}\,\overline{\Gamma^\mu(h)_{jk}} = \delta(g-h) \qquad \text{with} \qquad \int_G \delta(g-h)\,f(g)\,\mathrm{d}\mu(g) = f(h)\,.$$

6.8 Irreducible representations of SO(3)

For every $g \in \mathrm{SO}(3)$ exists an $X \in \mathfrak{so}(3)$ s.t. $g = \mathrm{e}^{\mathrm{i}X}$. Choose, e.g., the basis

$$J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}, \qquad J_2 = \begin{pmatrix} 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 \\ -\mathrm{i} & 0 & 0 \end{pmatrix}, \qquad J_3 = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

of $\mathfrak{so}(3)$ with

$$[J_j, J_k] = \mathrm{i} \sum_\ell \varepsilon_{jk\ell} J_\ell \,.$$

Then

$$R_{\vec{n}}(\psi) = \mathrm{e}^{-\mathrm{i}\psi \vec{n}\vec{J}} \qquad \text{where} \qquad \vec{n}\vec{J} = \sum_{j=1}^{3} n_j J_j$$

(rotation about axis $\vec{n}$ by angle ψ, cf. Section 6.5):

https://youtu.be/uPs-QSVt13s (6 min) (1)

From every representation of a Lie group we obtain (by taking derivatives) a representation of the corresponding Lie algebra (in terms of matrices). More precisely, with $g(t)$, $g(0) = e$, $\dot{g}(0) = \mathrm{i}X$ and a rep Γ of G define the derived rep $\mathrm{d}\Gamma$ of $\mathfrak{g}$ by

$$\mathrm{d}\Gamma(X) = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\Gamma\big(g(t)\big)\Big|_{t=0} \,.$$

From a representation of the Lie algebra $\mathfrak{so}(3)$ we obtain (by exponentiating) a representation of the group SO(3), if the global (topological) properties match those of SO(3).

Construction of reps of $\mathfrak{so}(3)$. The matrix (operator)

$$J^2 = \sum_{j=1}^{3} J_j^2$$

commutes with every $X \in \mathfrak{so}(3)$:

https://youtu.be/sxbtMVW2PJA (5 min) (2)

Remark: J^2 is not in the Lie algebra; it is a so-called Casimir operator and an element of the enveloping algebra (see later).

Consequences:

1. $[J^2, X] = 0 \;\forall\, X \in \mathfrak{so}(3)$ implies $[J^2, g] = 0 \;\forall\, g \in \mathrm{SO}(3)$. **Why?**
2. Consider a representation of SO(3). Now all this also holds for the representation matrices of g, X, and J^2.
3. If the representation is irreducible then according to Schur's Lemma (Theorem 4), the representation matrix of J^2 is a multiple of the identity matrix.

Next time we will construct all irreps of $\mathfrak{so}(3)$ in terms of simultaneous eigenvectors of the representation matrices of J^2 and one generator. After exponentiation these become irreps of SO(3) if the global properties are correct.

Groups and Representations

Instruction 20 for the preparation of the lecture on 5 July 2021

6.8 Irreducible representations of SO(3) (cont.)

Assume we are given a representation of SO(3).

Notation: We denote the representation matrices of g, X, J^2 also by g, X, J^2.

Construct irreducible subspaces (and thus irreps) as follows:

- Choose a suitable starting vector.
- Generate an irreducible basis by repeatedly applying the generators.

Suitable starting vector: Joint normalised eigenvector of J^2 and J_3 (**Why** can we choose it in this way?), in Dirac notation

$$J_3|m\rangle = m|m\rangle$$

We define $J_\pm = J_1 \pm \mathrm{i}J_2$. Then

$$[J_\pm, J_3] = \mp J_\pm \qquad \text{and thus} \qquad J_3 J_\pm|m\rangle = (m \pm 1)J_\pm|m\rangle\,, \tag{1}$$

https://youtu.be/4RE3ZFSPyGI (4 min)

i.e. either $J_\pm|m\rangle \propto |m \pm 1\rangle$ or $J_\pm|m\rangle = 0$.

Since the invariant subspace has to be finite dimensional the sequence

$$\dots, J_-|m\rangle, |m\rangle, J_+|m\rangle, J_+^2|m\rangle, \dots$$

has to terminate on both sides, say at $m = j$ and at $m = \ell$ with $j \geq \ell$,

$$\begin{aligned} J_3|j\rangle &= j|j\rangle\,, & J_3|\ell\rangle &= \ell|\ell\rangle\,, \\ J_+|j\rangle &= 0\,, & J_-|\ell\rangle &= 0\,. \end{aligned}$$

What is the dimension of this irreducible subspace?

We further have

$$J^2 = J_3^2 + J_-J_+ + J_3 \qquad \text{and} \qquad J^2 = J_3^2 + J_+J_- - J_3\,, \tag{2}$$

https://youtu.be/-qlcOB1JBmo (2 min)

and in particular (**why?**)

$$\begin{aligned} J^2|j\rangle &= (J_3^2 + J_3 + J_-J_+)|j\rangle = j(j+1)|j\rangle\,, \\ J^2|\ell\rangle &= (J_3^2 - J_3 + J_+J_-)|\ell\rangle = \ell(\ell-1)|\ell\rangle\,. \end{aligned}$$

Since both eigenvalues have to be identical (**why?**) we conclude that $\ell = -j$ (**why?**) and $j \geq 0$. Hence, we can label $\mathfrak{so}(3)$ irreps by $j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \dots$

- The dimension of irrep j is $2j + 1$.

From now on denote orthonormal basis vectors as $|jm\rangle$. Then

$$\begin{aligned} J^2|jm\rangle &= j(j+1)|jm\rangle \\ J_3|jm\rangle &= m|jm\rangle \\ J_\pm|jm\rangle &= [j(j+1) - m(m\pm 1)]^{1/2}|j, m\pm 1\rangle \end{aligned}$$

Verify the last identity by calculating the norm of $J_\pm|jm\rangle$.

Irreps of SO(3). Now we distinguish again between g and $\Gamma(g)$ and between X and $\mathrm{d}\Gamma(X)$. Denote by Γ^j the *potential* irrep of SO(3) carried by $\{|jm\rangle : m = -j \ldots, j\}$, i.e. the matrix elements are

$$\Gamma^j\left(\mathrm{e}^{-\mathrm{i}\psi\vec{n}\vec{J}}\right)_{mm'} = \langle jm|\,\mathrm{e}^{-\mathrm{i}\psi\,\mathrm{d}\Gamma(\vec{n}\vec{J})}\,|jm'\rangle\,.$$

Only for integer j does this define a representation of SO(3):

https://youtu.be/Cviw6oLYN68 (5 min) (3)

Irreps of SU(2). The Pauli matrices σ_1, σ_2, σ_3 (see Problem 32) form a basis of the Lie algebra $\mathfrak{su}(2)$ with

$$[\sigma_j, \sigma_k] = 2\mathrm{i}\sum_l \varepsilon_{jkl}\sigma_l\,,$$

i.e. the matrices $\sigma_k/2$ satisfy the same relations as the J_k, and thus $\mathfrak{su}(2) \cong \mathfrak{so}(3)$. Hence we also already know all irreps of $\mathfrak{su}(2)$. Since $\mathrm{SU}(2) = \exp(\mathrm{i}\mathfrak{su}(2))$ (cf. Problems 32 & 34), we get irreps of SU(2) for all $j \in \mathbb{N}_0/2$.

Determine the characters of all irreps of SO(3) and of all irreps of SU(2).

6.9 Remarks on some classical Lie groups

Definition: (adjoint representation)
Let G be a (matrix) Lie group with corresponding Lie algebra $\mathfrak{g}$, and let $g \in G$. The map $\mathrm{Ad} : g \mapsto \mathrm{Ad}_g$ with

$$\begin{aligned} \mathrm{Ad}_g : \mathfrak{g} &\to \mathfrak{g} \\ X &\mapsto gXg^{-1} = \mathrm{Ad}_g(X) \end{aligned}$$

is called adjoint representation of G (on $\mathfrak{g}$).

Remarks:

1. Ad is actually a representation. **Show this.**
2. We also define $\mathrm{Ad}_g(h) = ghg^{-1}$ for $h \in G$.
3. For $X \in \mathfrak{g}$ we further define $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$ by

$$\mathrm{ad}_X(Y) = \frac{1}{\mathrm{i}} \frac{\mathrm{d}}{\mathrm{d}t} \mathrm{Ad}_{\mathrm{e}^{\mathrm{i}Xt}}(Y)\bigg|_{t=0} = \frac{1}{\mathrm{i}} \frac{\mathrm{d}}{\mathrm{d}t} \left(\mathrm{e}^{\mathrm{i}Xt} Y \mathrm{e}^{-\mathrm{i}Xt}\right)\bigg|_{t=0} = [X, Y]\,.$$

Lemma 20. (Principal axis theorem for unitary matrices)
For every $g \in \mathrm{U}(n)$ there exists an $h \in \mathrm{U}(n)$ s.t. $h^\dagger g h$ is diagonal, in particular

$$g = h \begin{pmatrix} \mathrm{e}^{\mathrm{i}\varphi_1} & & 0 \\ & \ddots & \\ 0 & & \mathrm{e}^{\mathrm{i}\varphi_n} \end{pmatrix} h^\dagger$$

with real φ_j.

Proof: Reduce to the principal axis theorem for Hermitian matrices.
Let $M_\phi = \{g \in \mathrm{U}(n) \,:\, \mathrm{e}^{\mathrm{i}\phi} \text{ is not eigenvalue of } g\}$. Then

$$\begin{aligned} f_\phi : M_\phi &\to \mathbb{C}^{n\times n} \\ g &\mapsto \mathrm{i}(\mathrm{e}^{\mathrm{i}\phi} + g)(\mathrm{e}^{\mathrm{i}\phi} - g)^{-1} \end{aligned}$$

(generalised *Cayley transformation*) maps unitary g to Hermitian matrices $A = f_\phi(g)$:

https://youtu.be/TNfDf6iEpAo (5 min) (4)

Now there exists an $h \in \mathrm{U}(n)$ s.t. $h^\dagger A h = D$ is diagonal (principal axis theorem for Hermitian matrices). Furthermore, we can explicitly invert f_ϕ:

https://youtu.be/BZg9dikdumE (2 min) (5)

Finally, for given $g \in \mathrm{U}(n)$ choose ϕ s.t. $g \in M_\phi$, call $A = f_\phi(g)$, and choose $h \in \mathrm{U}(n)$ s.t. $h^\dagger A h = D$ is diagonal. Then h also diagonalises g:

$$h^\dagger g h = h^\dagger \mathrm{e}^{\mathrm{i}\phi}(A+\mathrm{i})^{-1} h h^\dagger (A - \mathrm{i}) h = \mathrm{e}^{\mathrm{i}\phi}(D+\mathrm{i})^{-1}(D-\mathrm{i})\,.$$

$\square$

Explain why the analogous result also holds for $g \in \mathrm{SU}(n) \subset \mathrm{U}(n)$, with $h \in \mathrm{SU}(n)$.

Groups and Representations

Instruction 21 for the preparation of the lecture on 7 July 2021

6.9 Remarks on some classical Lie groups (cont.)

Theorem 21. *For every $g \in \mathrm{U}(n)$ there exists an $X \in \mathfrak{u}(n)$ s.t. $g = \mathrm{e}^{\mathrm{i}X}$.*

Proof:

https://youtu.be/2SganmGZf7k (2 min) (1)

Remarks:

1. Similarly, for every $g \in \mathrm{SU}(n)$ there exists an $X \in \mathfrak{su}(n)$, s.t. $g = \mathrm{e}^{\mathrm{i}X}$. **Why?**
2. Similarly for $g \in \mathrm{SO}(2n)$: One first shows that there exists an $h \in \mathrm{SO}(2n)$ s.t.
$$g = h \begin{pmatrix} R_1 & & 0 \\ & \ddots & \\ 0 & & R_n \end{pmatrix} h^T$$
with $R_j \in \mathrm{SO}(2)$. **Why?** For $\mathrm{SO}(2n+1)$ the diagonal matrix has an additional row with an entry 1. Then also every $g \in \mathrm{SO}(n)$ can be written as $\mathrm{e}^{\mathrm{i}X}$ with $X \in \mathfrak{so}(n)$.
3. In all these cases we can in principle construct irreps using the same strategy as in Section 6.8 for SO(3) or SU(2): First construct irreducible representations of the Lie algebra and by exponentiation (potential) reps of the group.
4. The diagonal matrices which appear in this procedure are maximal abelian subgroups (so-called *maximal tori*) of the corresponding group.

6.10 More on Lie algebras and related topics

Definition: (representation of a Lie algebra)
Let $\mathfrak{g}$ be a Lie algebra and V a vector space. A representation ϕ is a linear map that assigns to each $X \in \mathfrak{g}$ a linear map $\phi(X) : V \to V$ s.t.

$$\phi(\mathrm{i}\underbrace{[X,Y]}_{\text{Lie bracket}}) = \underbrace{[\phi(X), \phi(Y)]}_{\text{commutator}} \qquad \forall\, X, Y \in \mathfrak{g}\,.$$

Remark: The i-decoration comes from our convention that $G = \exp(\mathrm{i}\mathfrak{g})$.

Examples:

1. $\mathrm{ad} : \mathfrak{g} \ni X \mapsto \mathrm{ad}_X$ with $\mathrm{ad}_X(Y) = [X, Y]$ defines a representation of $\mathfrak{g}$ on $\mathfrak{g}$:

https://youtu.be/wfkch23mM04 (5 min) (2)

2. From a rep Γ of a Lie group G we obtain (by differentiation) a rep $\mathrm{d}\Gamma$ of the Lie algebra $\mathfrak{g}$,
$$\mathrm{d}\Gamma(X) = \frac{1}{\mathrm{i}}\,\frac{\mathrm{d}}{\mathrm{d}t}\Gamma(\mathrm{e}^{\mathrm{i}Xt})\bigg|_{t=0}\,.$$

Definition: (enveloping algebra)
Let $\mathfrak{g}$ be a Lie algebra with basis $\{X_j\}$. The enveloping algebra $E(\mathfrak{g})$ consists of formal polynomials in the generators

$$\sum_j a_j(\mathrm{i}X_j) + \sum_{jk} b_{jk}(\mathrm{i}X_j)(\mathrm{i}X_k) + \sum_{jkl} c_{jkl}(\mathrm{i}X_j)(\mathrm{i}X_k)(\mathrm{i}X_l) + \dots\,, \qquad a_j, b_{jk}, c_{jkl} \in \mathbb{R}\,,$$

where $\mathrm{i}X_j\mathrm{i}X_k$ and $\mathrm{i}X_k\mathrm{i}X_j + \mathrm{i}X_l$ have to be identified if $[\mathrm{i}X_j, \mathrm{i}X_k] = \mathrm{i}X_l$.

Remarks:

1. A representation ϕ of a Lie algebra then also induces a representation of the enveloping algebra, whereby the formal products and sums become matrix products and matrix sums.
2. A basis of the enveloping algebra is, e.g., given by those monomials in the generators for which the indices are non-decreasing from left to right:

https://youtu.be/tydgmSEW18I (2 min) (3)

Definition: (Casimir operator)
$C \in E(\mathfrak{g})$ is called Casimir operator if C commutes with all elements of the enveloping algebra, i.e. if
$$[C, A] = 0 \quad \forall\ A \in E(\mathfrak{g})\,.$$

Example: $J^2 := J_1^2 + J_2^2 + J_3^2$ for SO(3) (cf. Section 6.8).

Remarks:

1. In particular a Casimir operator commutes with all $X \in \mathfrak{g} \subseteq E(\mathfrak{g})$.
2. This implies $\mathrm{e}^{\mathrm{i}X} C \mathrm{e}^{-\mathrm{i}X} = C\ \forall\ X \in \mathfrak{g}$, i.e. in the cases of Sections 6.8 and 6.9, where $G = \exp(\mathrm{i}\mathfrak{g})$, we immediately conclude $gCg^{-1} = C\ \forall\ g \in G$.
3. $gCg^{-1} = C\ \forall\ g \in G$ is even true more generally, since one can show:
   - $\exp(\mathrm{i}\mathfrak{g})$ always contains a neighbourhood of the identity in G.
   - By taking (finite) products $\mathrm{e}^{\mathrm{i}X}\mathrm{e}^{\mathrm{i}Y}\mathrm{e}^{\mathrm{i}Z}\dots$ one reaches all $g \in G_0$, the connected component of the identity.
   - If G is connected, then for representations (of the Lie group, the Lie algebra and the enveloping algebra) we thus have $[\mathrm{d}\Gamma(C), \Gamma(g)] = 0\ \forall\ g \in G$, and according to Schur's Lemma (Theorem 4) it follows that for irreps $\mathrm{d}\Gamma(C)$ is a scalar multiple of $\mathbb{1}$.

Universität Tübingen, Fachbereich Mathematik
Dr. Stefan Keppeler Sommersemester 2021

Groups and Representations

Instruction 22 for the preparation of the lecture on 12 July 2021

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7 Tensor method for constructing irreps of GL(N) and subgroups

7.1 Setting

In the following let V be complex vector space with $\dim V = N$, i.e. $V \cong \mathbb{C}^N$.

Define $V^{\otimes n} = \underbrace{V \otimes \cdots \otimes V}_{n \text{ factors}}$. Form tensor products from $|v_j\rangle \in V$, $j = 1, \dots, n$:

$$\bigotimes_{j=1}^{n} |v_j\rangle = |v_1\rangle \otimes |v_2\rangle \otimes \cdots \otimes |v_n\rangle \in V^{\otimes n} .$$

General $|v\rangle \in V^{\otimes n}$ are linear combinations of tensor products, called *tensors of rank n*.

$V^{\otimes n}$ carries reps Γ of GL(N) and D of S_n

$$\Gamma(g) \bigotimes_{j=1}^{n} |v_j\rangle = \bigotimes_{j=1}^{n} \gamma(g)|v_j\rangle , \qquad \text{with } \gamma(g) = g \text{ (defining rep)},$$

$$D(p)\big(|v_1\rangle \otimes |v_2\rangle \otimes \cdots \otimes |v_n\rangle\big) = |v_{p^{-1}(1)}\rangle \otimes |v_{p^{-1}(2)}\rangle \otimes \cdots \otimes |v_{p^{-1}(n)}\rangle ,$$

everything continued by linearity; D also extends to a rep of $\mathcal{A}(S_n)$.

Convince yourself that D is a rep.

These reps commute, i.e.

$$\Gamma(g)D(p)|v\rangle = D(p)\Gamma(g)|v\rangle \qquad \forall\, p \in S_n\, , \ \forall\, g \in \mathrm{GL}(N)\, , \ \forall\, |v\rangle \in V^{\otimes n} ,$$

and even $\forall\, p \in \mathcal{A}(S_n)$.

Notation: Form now on, we omit Γ and D, e.g. we write $gp|v\rangle = pg|v\rangle$.

How does $p \in S_n$ act on an arbitrary $|x\rangle \in V^{\otimes n}$?

https://youtu.be/EEEq-bCuc5c (3 min) (1)

7.2 Decomposition of $V^{\otimes n}$ into irreducible invariant subspaces with respect to S_n and $\mathrm{GL}(N)$

7.2.1 Symmetry classes

Let Θ_λ^p be a Young tableau, e_λ^p the corresponding Young operator, and $L_\lambda = \mathcal{A}(S_n)e_\lambda$ the minimal left ideal generated by e_λ

In the following we will see:

For fixed $|v\rangle \in V^{\otimes n}$ the subspace

$$L_\lambda|v\rangle = \mathcal{A}(S_n)e_\lambda|v\rangle$$

(if non-empty) is invariant and irreducible w.r.t. S_n.

The subspace

$$e_\lambda^p V^{\otimes n}$$

is invariant and irreducible w.r.t. $\mathrm{GL}(N)$.

Then we will be able to choose a basis $\{|\lambda, \alpha, a\rangle\}$ of $V^{\otimes n}$ s.t.
λ lables the so-called symmetry class, given by a Young diagram,
α labels the irreducible invariant subspaces w.r.t. S_n,
a labels the irreducible invariant subspaces w.r.t. $\mathrm{GL}(N)$.

Lemma 22. *For fixed $|\alpha\rangle \in V^{\otimes n}$ the subspace $T_\lambda(\alpha) = L_\lambda|\alpha\rangle$ is either empty or*

(i) *$T_\lambda(\alpha)$ is invariant and irreducible under S_n and*

(ii) *the S_n irrep carried by $T_\lambda(\alpha)$ is given by the irrep carried by L_λ.*

Proof:

https://youtu.be/Nv1AecrF2vE (6 min) (2)

7.2.2 Totally symmetric and totally anti-symmetric tensors

Let $\lambda = \mathrm{s} = \boxed{}\boxed{}\boxed{\cdots}\boxed{}$, i.e. $e_\mathrm{s} = s$ is the total symmetriser of S_n, L_s is one-dimensional.

$\Rightarrow$ For given $|\alpha\rangle$ the subspace $T_\mathrm{s}(\alpha)$ is one-dimensional, $T_\mathrm{s}(\alpha) = \mathrm{span}(e_s|\alpha\rangle)$.

These tensors are *totally symmmetric* (in all indices).

Each $T_\mathrm{s}(\alpha)$ carries the trivial representation of S_n.

Example: $n = 3,\ N = 2$

https://youtu.be/Crhbo74Jj0k (5 min) (3)

We denote the space spanned by the tensors of symmetry class s by T'_s.

Totally anti-symmetric tensors exist only for $n \leq N$,

$\lambda = \mathrm{a} = \begin{array}{|c|}\hline \\ \hline \\ \hline \vdots \\ \hline \\ \hline \end{array}\,,$ since for $n > N$ every basis vector contains at least two identical indices, anti-symmetrisation yields zero.

The S_n irrep carried by $T_\mathrm{a}(\alpha)$ is sgn.

Example: $n = 2,\ N \geq 2$

https://youtu.be/sX_vkzbmiiQ (2 min) (4)

Construct all totally symmetric tensors for $n = 2$ and arbitrary N. How many are there?

Groups and Representations

Instruction 23 for the preparation of the lecture on 14 July 2021

7.2.3 Tensors with mixed symmetry

Example: Consider tensors of rank $n = 3$ in $N = 2$ dimensions, and choose

$$\Theta_\kappa = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \qquad \text{with} \qquad e_\kappa = e + (12) - (13) - (132)\,.$$

From Section 5.3 we know: $L_\kappa = \mathrm{span}(e_\kappa, (23)e_\kappa)$

▶ First we choose $|\alpha\rangle = |112\rangle$:

$$\begin{aligned} e_\kappa|112\rangle &= 2|112\rangle - |211\rangle - |121\rangle \qquad =: |\kappa,1,1\rangle\,, \\ (23)e_\kappa|112\rangle &= 2|121\rangle - |211\rangle - |112\rangle \qquad =: |\kappa,1,2\rangle\,. \end{aligned}$$

https://youtu.be/saVR889k6qA (3 min)

$\Rightarrow T_\kappa(1) = \mathcal{A}(S_3)e_\kappa|112\rangle = \mathrm{span}(|\kappa,1,1\rangle, |\kappa,1,2\rangle)$ is invariant and irreducible under S_3.

▶ Then we choose $|\alpha\rangle = |221\rangle$:

$$\begin{aligned} e_\kappa|221\rangle &= 2|221\rangle - |122\rangle - |212\rangle] =: |\kappa,2,1\rangle\,, \\ (23)e_\kappa|221\rangle &= 2|212\rangle - |122\rangle - |221\rangle] =: |\kappa,2,2\rangle\,, \end{aligned}$$

$\Rightarrow T_\kappa(2) = \mathcal{A}(S_3)e_\kappa|221\rangle = \mathrm{span}(|\kappa,2,1\rangle, |\kappa,2,2\rangle)$ is invariant and irreducible under S_3.

▶ $|\kappa,1,1\rangle$ and $|\kappa,2,1\rangle$ span the 2-dimensional subspace $T'_\kappa(1) = e_\kappa V^{\otimes 3}$.

(i) $T'_\kappa(1)$ is invariant under $\mathrm{GL}(2)$, since $gp = pg\ \forall\, g \in \mathrm{GL}(2)$ and $\forall\, p \in S_3$ implies

$$g e_\kappa|v\rangle = e_\kappa g|v\rangle \in T'_\kappa(1)\,.$$

This argument requires neither $n = 3$ nor $N = 2$, nor $\lambda = \kappa$ – it is true in general!

(ii) $T'_\kappa(1)$ is irreducible under $\mathrm{GL}(2)$.

Proof: We explicitly construct the representation matrices for $g \in \mathrm{GL}(2)$:

$$\Gamma^\kappa(g) = \det g \begin{pmatrix} g_{11} & -g_{12} \\ -g_{21} & g_{22} \end{pmatrix} \qquad \text{https://youtu.be/fN3r9Ja6wkU (10 min)} \qquad (1)$$

Why does this prove irreducibility of Γ^κ?

▶ Similarly, $|\kappa,1,2\rangle$ and $|\kappa,2,2\rangle$ span $T'_\kappa(2) = e_\kappa^{(23)}V^{\otimes 3}$, which is also invariant and irreducible under $\mathrm{GL}(2)$ and carries a rep that is equivalent to Γ^κ.

▶ The direct sum $T'_\kappa(1) \oplus T'_\kappa(2)$ contains all tensors of symmetry class $\kappa = \begin{array}{|c|c|}\hline \ & \ \\ \hline \ \\ \cline{1-1}\end{array}$.

Summary: Complete reduction of the 8-dimensional space $V^{\otimes 3}$:
(recall that $\Theta_\mathrm{s} = \ydiagram{3}$ and $\Theta_\kappa = \ydiagram{2,1}$)

$$
\begin{aligned}
V^{\otimes 3} &= \underbrace{T_\mathrm{s}(1) \oplus T_\mathrm{s}(2) \oplus T_\mathrm{s}(3) \oplus T_\mathrm{s}(4)} \quad \oplus \underbrace{T_\kappa(1) \oplus T_\kappa(2)} && \leftarrow \text{invariant under } S_3 \\
&= \qquad\qquad T'_\mathrm{s} \qquad\qquad\quad \oplus \overbrace{T'_\kappa(1) \oplus T'_\kappa(2)} && \leftarrow \text{invariant under } \mathrm{GL}(2)
\end{aligned}
$$

T'_s carries a 4-dimensional irrep of GL(2); under S_3 it is the direct sum of 4 one-dimensional subspaces, each carrying the trivial rep.

As a convenient basis for $V^{\otimes 3}$ we can choose:

- the 4 totally symmetric tensors from Section 7.2.2, and
- the 4 tensors $|\kappa, \alpha, a\rangle$ with $\alpha = 1, 2$ and $a = 1, 2$.

7.2.4 Complete reduction of $V^{\otimes n}$

The observations and results of the preceding sections generalise. We will look at this together in our live session.

7.2.5 Dimensions of the GL(N) irreps

Essentially, we already know the dimensions of the GL(N) irreps: To each Young diagram λ corresponds an S_n irrep D^λ and a GL(N) irrep Γ^λ. For the S_n irreps we can determine dimensions and multiplicities (within $V^{\otimes n}$) using the methods of Sections 4.3.1 and 5.5. According to the construction in Sections 7.2.1–4 the multiplicity of D^λ is equal to the dimension of Γ^λ and vice versa. Determining the dimensions in this way can be tedious, and there are several other algorithms and formulae. We will speak about the following two in our live session:

- ▶ **Graphical rule:** The dimension of the GL(N) irrep corresponding to the Young diagram λ is given by the number of *semi-standard* Young tableaux Θ_λ. In semi-standard Young tableaux numbers need not increase in rows but must only be non-deceasing.

  Example: For $N = 2$ we find

  $$\dim \Gamma^{\ydiagram{2,1}} = 2 \qquad \text{and} \qquad \dim \Gamma^{\ydiagram{3}} = 4\,,$$

  since the allowed choices are

  $$\ytableaushort{11,2}\,,\ \ytableaushort{12,2} \qquad \text{and} \qquad \ytableaushort{111}\,,\ \ytableaushort{112}\,,\ \ytableaushort{122}\,,\ \ytableaushort{222}\,.$$

  Determine the corresponding dimensions for $N = 3$.

- ▶ **Hook length formula:**

  $$\dim(\Gamma^\lambda) = \prod_{ij} \frac{N + j - i}{\underset{\substack{\uparrow \\ \text{hook length of box } i,j}}{h_{ij}}} \qquad \begin{pmatrix} \text{product over all boxes of } \lambda \\ i = \text{row},\ j = \text{column index} \end{pmatrix}$$

Universität Tübingen, Fachbereich Mathematik — Sommersemester 2021
Dr. Stefan Keppeler

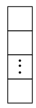
Groups and Representations

Instruction 24 for the preparation of the lecture on 19 July 2021

7.3 Irreps of U(N) and SU(N)

The GL(N) irreps from Section 7.2 restrict to representations of subgroups, which do not need to be irreducible. They are, however, irreducible for U(N) and SU(N) but in general not for O(N) and SO(N).

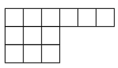
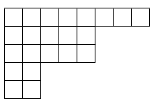
U(N) and SU(N)	https://youtu.be/WM6vX88PKG4 (4 min)	(1)
O(N) and SO(N)	https://youtu.be/_-ooGDPg204 (4 min)	(2)

Show that the GL(N) irrep corresponding to the Young diagram **a** =  with N rows is given by the determinant:

- ▶ First recall that for vectors $|i_1, \dots, i_N\rangle$ contributing to $e_{\mathrm{a}} g|\alpha\rangle$ all i_k are different.
- ▶ Write these vectors as $p|1, \dots, N\rangle$ with a permutation p.
- ▶ Then calculate $e_{\mathrm{a}} g|1, \dots, N\rangle$ for $g \in \mathrm{GL}(N)$.

Which irrep corresponds to **a** if we replace GL(N) by the subgroup SU(N)?

In the exercises we will show that the SU(N) irreps corresponding to the Young diagrams (with row lenghts) $(\lambda_1, \dots, \lambda_N)$ and $(\lambda_1{+}k, \dots, \lambda_N{+}k)$ are equivalent, e.g.

 and  for $N = 5$ and $k = 2$.

For SU(2), except for the trivial rep, all irreps can be labelled by one-row Young diagrams. **What** are the corresponding dimensions?

7.4 Reducing tensor products in terms of Young diagrams

Goal: Given two irreps Γ^λ and $\Gamma^{\lambda'}$ of GL(N), U(N) or SU(N) with Young diagrams λ and λ' find the complete reduction of the product rep $\Gamma^\lambda \otimes \Gamma^{\lambda'}$.

Examples and observations:

$\square^{\otimes 2}$, $\square^{\otimes 3}$, $\square^{\otimes 4}$ https://youtu.be/FLPJbunjr9U (11 min) (3)

Closer inspections leads to the Littlewood-Richardson rule (which we won't prove):

1. Write the number i in all boxes of row i of λ'.
2. Add the boxes of λ' to λ, first the 1s, then the 2s etc. adhering to the following rules:

(a) In each step the resulting diagram has to be a valid Young diagram and must not have more than N rows.

(b) No number may appear more than once in the same column.

(c) When reading the numbers row-wise from *right to left* beginning with the first row, then the second etc., and terminating this sequence at any point, there must never be more is than $(i-1)$s.

3. For SU(N) columns with N boxes can be omitted.

Always check your result by comparing dimensions on both sides of the equation.

Example:

$$\ydiagram{2,2} \otimes \ydiagram{2,1} \quad \text{for SU(3)} \qquad \texttt{https://youtu.be/xrze6-yRWTI}\ (10\,\text{min}) \tag{4}$$

Reduce $\ydiagram{2,1} \otimes \ydiagram{2,1}$ for SU(3).

7.5 Complex conjugate representations

Observation: Sometimes $\dim \Gamma^{\lambda} = \dim \Gamma^{\lambda'}$ for $\lambda \neq \lambda'$. This may be "accidental" but often it can be understood systematically in terms of the following construction.

Example: Consider $\ydiagram{1,1}$ for $N = 3$.

Basis tensors: (anti-symmetric tensors of rank 2 in 3 dimensions)

$$|23\rangle - |32\rangle\,, \qquad |31\rangle - |13\rangle\,, \qquad |12\rangle - |21\rangle\,.$$

Action of GL(3), e.g.

$$\begin{aligned}
g(|12\rangle - |21\rangle) &= |ij\rangle(g_{i1}g_{j2} - g_{i2}g_{j1}) \\
&= \underbrace{|23\rangle(g_{21}g_{32} - g_{22}g_{31}) + |32\rangle(g_{31}g_{22} - g_{32}g_{21})}_{=(|23\rangle - |32\rangle)\,\det\begin{pmatrix} g_{21} & g_{22} \\ g_{31} & g_{32}\end{pmatrix}} \\
&\quad + \underbrace{|31\rangle(g_{31}g_{12} - g_{32}g_{11}) + |13\rangle(g_{11}g_{32} - g_{12}g_{31})}_{=(|31\rangle - |13\rangle)\,(-1)\det\begin{pmatrix} g_{11} & g_{12} \\ g_{31} & g_{32}\end{pmatrix}} \\
&\quad + \underbrace{|12\rangle(g_{11}g_{22} - g_{12}g_{21}) + |21\rangle(g_{21}g_{12} - g_{22}g_{11})}_{=(|12\rangle - |21\rangle)\,\det\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22}\end{pmatrix}}\,,
\end{aligned}$$

similarly for the other two basis elements. We find

$$\Gamma^{\ydiagram{1,1}}(g) = \begin{pmatrix}
\det\begin{pmatrix} g_{22} & g_{23} \\ g_{32} & g_{33}\end{pmatrix} & (-1)\det\begin{pmatrix} g_{21} & g_{23} \\ g_{31} & g_{33}\end{pmatrix} & \det\begin{pmatrix} g_{21} & g_{22} \\ g_{31} & g_{32}\end{pmatrix} \\
(-1)\det\begin{pmatrix} g_{12} & g_{13} \\ g_{32} & g_{33}\end{pmatrix} & \det\begin{pmatrix} g_{11} & g_{13} \\ g_{31} & g_{33}\end{pmatrix} & (-1)\det\begin{pmatrix} g_{11} & g_{12} \\ g_{31} & g_{32}\end{pmatrix} \\
\det\begin{pmatrix} g_{12} & g_{13} \\ g_{21} & g_{23}\end{pmatrix} & (-1)\det\begin{pmatrix} g_{11} & g_{13} \\ g_{21} & g_{23}\end{pmatrix} & \det\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22}\end{pmatrix}
\end{pmatrix} = \mathrm{adj}(g)^T\,,$$

with the adjunct matrix $\mathrm{adj}(g)$. According to Cramer's rule $g^{-1} = \dfrac{\mathrm{adj}(g)}{\det g}$, i.e.

$$\Gamma^{\tiny\yng(1,1)}(g) = \det g \cdot (g^{-1})^T \,.$$

Remark: This is true for arbitrary $N > 2$ and the Young diagram $\begin{array}{|c|}\hline \, \\ \hline \vdots \\ \hline \, \\ \hline\end{array}$ ($N-1$ boxes).

For SU(3) we have $\det g = 1$ and $g^{-1} = g^\dagger$, i.e. $\Gamma^{\tiny\yng(1,1)}(g) = \overline{g}$. We write $\yng(1,1) = \overline{\yng(1)}$ and also put a $\overline{\text{bar}}$ over the dimension

For GL(N), besides the defining rep g also $(g^{-1})^T$, $\overline{g}$ and $\overline{(g^{-1})^T}$ are N-dimensional irreps, in general non-equivalent.

For SU(N), due to $g^\dagger = g^{-1}$, we have

$$(g^{-1})^T = \overline{g} \qquad \text{and} \qquad \overline{(g^{-1})^T} = g \,,$$

i.e. at most two of the four irreps are non-equivalent. For SU(2), even g and $\overline{g}$ are equivalent, see Problem 40; for $N \geq 3$ they are are non-equivalent. In terms of Young diagrams we obtain the complex conjugate irrep by means of a simple procedure which we will study in the live session.