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Inequalities

17. November 2022

Part 1: Introduction to Inequalities

We first consider the following two fundamental inequalities:

- If x is a real number, then
$$\boxed{x^2 \geq 0} \tag{Ineq 1}$$
Moreover, equality holds if, and only if, $x = 0$.
- If $a, b \in \mathbb{R}$, then:
$$\boxed{a^2 + b^2 \geq 2ab} \tag{Ineq 2}$$
Moreover, equality holds if, and only if, $a = b$.

Using only these inequalities, we can solve the following problems:

Problem 1. Prove that, if $a, b, c \in \mathbb{R}$, then $a^2 + b^2 + c^2 \geq ab + bc + ca$.

Problem 2. Find all numbers $a, b, c, d \in \mathbb{R}$ such that $a^2 + b^2 + c^2 + d^2 = a(b + c + d)$.

Problem 3. If $a, b, c \in \mathbb{R}$ are positive and such that $a^2 + b^2 + c^2 = 1$, find the minimal value of
$$\frac{a^2b^2}{c^2} + \frac{b^2c^2}{a^2} + \frac{c^2a^2}{b^2} \,. \tag{1}$$

Problem 4. If $a, b, c \in \mathbb{R}$ are positive and such that
$$\frac{a^2}{1+a^2} + \frac{b^2}{1+b^2} + \frac{c^2}{1+c^2} = 1 \,, \tag{2}$$
prove that
$$abc \leq \frac{1}{2\sqrt{2}} \,. \tag{3}$$

Problem 5 (Nesbit's inequality). If $a, b, c \in \mathbb{R}$ are positive, prove that:
$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2} \,. \tag{4}$$

Arithmetic-Geometric Inequality

Let $x_1, x_2, \ldots x_n$ be positive real numbers. The following inequalities hold:

$$\boxed{0 < \underbrace{\frac{n}{\frac{1}{x_1} + \ldots + \frac{1}{x_n}}}_{\text{Harmonic mean}} \leq \underbrace{\sqrt[n]{x_1 \ldots x_n}}_{\text{Geometric mean}} \leq \underbrace{\frac{x_1 + \ldots + x_n}{n}}_{\text{Arithmetic mean}} \leq \underbrace{\sqrt{\frac{x_1^2 + \ldots + x_n^2}{n}}}_{\text{Quadratic mean}}} \tag{Ineq 3}$$

The most-frequently used one is the arithmetic-geometric inequality. Equality holds in all the previous inequalities if, and only if, $x_1 = \ldots = x_n$.

Using this and the previous inequalities, we can solve the following problems:

Problem 6. Prove that, if $a, b \in \mathbb{R}$ are positive, then $2a^3 + b^3 \geq 3a^2 b$.

Problem 7. Prove that, if $a, b, c \in \mathbb{R}$ are positive, then the following holds:

$$\frac{a^3}{a^2 + ab + b^2} + \frac{b^3}{b^2 + bc + c^2} + \frac{c^3}{c^2 + ca + a^2} \geq \frac{a + b + c}{3}, \tag{5}$$

Hölder's inequality

Let $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in \mathbb{R}^n$ or $\mathbb{C}^n$. We have

$$\sum_{i=1}^{n} |x_i y_i| \leq \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} \left(\sum_{i=1}^{n} |y_i|^q \right)^{1/q}, \tag{6}$$

for p, q satisfying $1 = \frac{1}{p} + \frac{1}{q}$. Equality holds if, and only if, there is a constant λ such that $x_i = \lambda y_i$ for all $i = 1, \ldots, n$.

A particular case of this is Cauchy-Schwarz inequality:

$$\sum_{i=1}^{n} |x_i y_i| \leq \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2} \left(\sum_{i=1}^{n} |y_i|^2 \right)^{1/2}. \tag{7}$$

Young's inequality

Let a, b, p and q be positive real numbers with $\frac{1}{p} + \frac{1}{q} = 1$. Then, we have

$$\frac{a^p}{p} + \frac{b^q}{q} \geq ab\,. \tag{8}$$

Equality holds if, and only if, $a^p = b^q$.

Minkowski's inequality

Let $x_1, \ldots, x_n, y_1, \ldots, y_n$ and $m_1, \ldots, m_n$ be three sequences of positive real numbers and $p > 1$. Then, the following holds

$$\left(\sum_{i=1}^{n}(x_i + y_i)^p m_i\right)^{1/p} \leq \left(\sum_{i=1}^{n} x_i^p m_i\right)^{1/p} + \left(\sum_{i=1}^{n} y_i^p m_i\right)^{1/p} . \tag{9}$$

The equality holds if, and only if, there is a constant λ such that $x_i = \lambda y_i$ for all $i = 1, \ldots, n$.

Jensen's inequality

Let $f : [a, b] \to \mathbb{R}$ be a convex function and $\alpha_1, \ldots, \alpha_n$ a sequence of positive numbers such that $\alpha_1 + \ldots + \alpha_n = 1$. Then, for any sequence $x_1, \ldots, x_n \in [a, b]$, we have

$$f\left(\sum_{i=1}^{n} \alpha_i x_i\right) \leq \sum_{i=1}^{n} \alpha_i f(x_i) . \tag{10}$$

If f is concave, the previous inequality is reversed.

Chebyshev's inequality

Let $a_1 \geq \ldots \geq a_n$ and $b_1 \geq \ldots \geq b_n$ be real numbers. Then,

$$n\sum_{i=1}^{n} a_i b_i \geq \left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} b_i\right) \geq n\sum_{i=1}^{n} a_i b_{n+1-i} . \tag{11}$$

Both inequalities are equalities if, and only if, $a_1 = \ldots = a_n$ or $b_1 = \ldots = b_n$.