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In these notes, I assume some basic knowledge about sets and functions. For a review of some notions on these topics, I refer the reader to this link for sets and this link for functions.

Part 1: Introduction to Counting

We will start counting elements of sets. For that, we will use the following widely used fact in combinatorics: Two sets have the same number of elements if, and only if, there is a bijection between them. To explore this idea, let us solve the following two examples.

Example 1. In how many ways can we distribute 15 identical apples to 4 distinct students? Not all students have to get an apple.

Example 2. Determine the number of subsets of $\{1, 2, 3, \dots, 50\}$ whose sum of elements is larger than or equal to 638.

Part 2: Generating Functions

Given a sequence $\{a_n\}_{n=0}^{\infty}$, the generating function corresponding to the sequence $\{a_n\}_{n=0}^{\infty}$ is the function defined as:

$$\boxed{F(X) = \sum_{n=0}^{\infty} a_n X^n} \tag{1}$$

Example 3. Find the generating function for the sequence $\{a_n\}_{n=0}^{\infty}$ defined as $a_n = 1$ for all $n \geq 0$.

Example 4. Find the generating function for the sequence $\{a_n\}_{n=0}^{\infty}$ defined as $a_n = n$ for all $n \geq 0$.

The following theorem will be useful in some exercises below:

Theorem. If $a \in \mathbb{R}$, then:

$$(1+X)^a = \sum_{k=0}^{\infty} \binom{a}{k} X^k$$

Next, we are going to use generating functions to solve problems in combinatorics. For that, let us consider the following example.

Example 5. In how many ways can we distribute 15 identical apples to 5 distinct students? Not all students have to get an apple.

Example 6. Let n be a positive integer. Assume that N_k is the number of pairs (a, b) of non-negative integers such that $ka + (k+1)b = n + 1 - k$. Find $N_1 + N_2 + \ldots + N_{n+1}$.

Finally, we can also use the exponential generating function to solve combinatorics problems. Given a sequence $\{a_n\}_{n=0}^{\infty}$, the exponential generating function corresponding to it is the function defined as:

$$\boxed{F(X) = \sum_{n=0}^{\infty} \frac{a_n}{n!} X^n} \tag{2}$$

Example 7. Suppose that there are p different kinds of objects, each in infinite supply. Let a_k be the number of different ways to arrange k of these objects in a line. Find a_k explicitly by using exponential generating functions.

For a more thorough introduction to the theory of generating functions, I refer the reader to this link.

Application to Recursive Equations

Here, we are going to use generating functions to solve problems related to recursive equations.

Problem 1. Let a_n be a sequence given by $a_0 = 0$ and $a_{n+1} = 2a_n + 1$ for $n \geq 0$. Find the general term of the sequence a_n.

Problem 2. Let a_n be a sequence given by $a_0 = 1$ and $a_{n+1} = 2a_n + n$ for $n \geq 0$. Find the general term of the sequence a_n.

Problem 3. Find the general term of the Fibonacci sequence.